

GENERIC 1D-INTERLEAVINGS

Benedikt Fluhr

Rheinische Friedrich-Wilhelms-Universität Bonn

Introduction

Suppose $f: X \rightarrow \mathbb{R}$ and $g: Y \rightarrow \mathbb{R}$ are continuous functions. Then we may ask the following two questions:

- Is there a homeomorphism $\varphi: X \rightarrow Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ f \searrow & & \nearrow g \\ & \mathbb{R} & \end{array}$$

commutes? (In the following we call φ and any other continuous map that fits into the above diagram a *homomorphism* from f to g . This augments the class of real-valued continuous functions with the structure of a category, the *category of \mathbb{R} -spaces* for short.)

- Is there a homeomorphism $\varphi: X \rightarrow Y$ and a real number r such that

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ f \downarrow & & \downarrow g \\ \mathbb{R} & \xrightarrow{(r+_)} & \mathbb{R} \end{array}$$

commutes? (In other words $r + f(p) = g(\varphi(p))$ for all $p \in X$.)

If the answer to either of the above questions is no, we may still ask a weaker question. Instead of commutativity of the above diagrams, we only ask for commutativity up to ε for some $\varepsilon \geq 0$. For $\varepsilon \geq 0$ we have the two questions:

- Is there a homeomorphism $\varphi: X \rightarrow Y$ such that for all $p \in X$ the estimates $-\varepsilon \leq f(p) - g(\varphi(p)) \leq \varepsilon$ hold?
- Is there a homeomorphism $\varphi: X \rightarrow Y$ and a real number r such that for all $p \in X$ the estimates $-\varepsilon \leq r + f(p) - g(\varphi(p)) \leq \varepsilon$ hold?

Now both these questions depend on the parameter $\varepsilon \geq 0$ and for either question we may ask how large does ε need to be in order for the answer to be *yes*.

Definition. Let $M(f, g)$ be the infimum of all $\varepsilon \geq 0$ such that the answer to the first question is affirmative. We name this the *absolute distance* of f and g .

And let $\mu(f, g)$ be the infimum of all $\varepsilon \geq 0$ such that the answer to the second question is affirmative. We name this the *relative distance* of f and g .

In the very first question we ask whether f and g are isomorphic in the category of \mathbb{R} -spaces. Thus we may apply a functor F from the category of \mathbb{R} -spaces to a category in which it is easier to decide whether two objects are isomorphic or not. If $F(f)$ and $F(g)$ are isomorphic, then we still don't know. But if $F(f)$ and $F(g)$ are not isomorphic, then we know the answer is *no*. Similarly we may never learn the absolute or relative distance of f and g but we may try to find lower bounds to either value. To this end functors from the category of \mathbb{R} -spaces may be useful, but this concept alone doesn't solve our problem. In the following we describe a framework for *enhancing* functors from the category of \mathbb{R} -spaces to obtain lower bounds for $M(f, g)$ and $\mu(f, g)$. On several aspects this borrows from [BdS14]. The work presented here is supported by the DFG-project DFG (KL 655/19) (D-A-CH) and a part of my master's thesis supervised by Professor Rolf Klein. His advice has always been very useful and relevant to my work.

Monoidal Posets for 1D-Interleavings

Definition. Let $D := \{(a, b) \in [-\infty, \infty) \times (-\infty, \infty] \mid a \leq b\}$ then D is a monoid by component-wise addition. Moreover we set $\mathbf{o} := (0, 0)$ and we define a partial order \preceq on D by

$$(x, y) \preceq (x', y') \text{ if and only if } x \geq x' \text{ and } y \leq y'.$$

A set augmented with a partial ordering and a monoid structure like D , we call a *monoidal poset*. Moreover $D \times D$ is a monoidal poset with the product ordering and component-wise addition. With some abuse of notation we write $(a, b; c, d)$ for points $((a, b), (c, d)) \in D \times D$. Now we define three monoidal sub-posets of $D \times D$.

Definition. We set

- $\mathcal{D} := \{(a, b; c, d) \in D \times D \mid a + c \leq 0 \leq b + d\}$,
- $\nabla := \{(a, b; c, d) \in \mathcal{D} \mid c + b = 0 = a + d\}$, and
- $\blacktriangledown := \{(a, b; c, d) \in \nabla \mid a + b = 0\}$.

Both the inclusion of ∇ into \mathcal{D} and the inclusion of \blacktriangledown into ∇ each have a lower adjoint in the sense of monotone Galois connections.

Definition. Let $\delta: \mathcal{D} \rightarrow \nabla$ be the lower adjoint of $\nabla \subset \mathcal{D}$ and let $\gamma: \nabla \rightarrow \blacktriangledown$ be the lower adjoint of $\blacktriangledown \subset \nabla$.

We note that both δ and γ are sub-linear, i.e. oplax monoidal as functors. We use these maps to describe two weightings on \mathcal{D} .

Definition. We set $\varepsilon: \nabla \rightarrow [0, \infty]$, $(a, b; c, d) \mapsto \frac{1}{2}(b + d)$.

We note that ε is a (strict) homomorphism of monoidal posets. Moreover the restriction $\varepsilon|_{\blacktriangledown}$ is an isomorphism. The compositions $\varepsilon \circ \gamma \circ \delta$ and $\varepsilon \circ \delta$ yield two different weightings on \mathcal{D} . It turns out $(\varepsilon \circ \gamma \circ \delta)((a, b; c, d)) = \max\{-a, b, -c, d\}$ and $(\varepsilon \circ \delta)((a, b; c, d)) = \frac{1}{2}(\max\{-c, b\} + \max\{-a, d\})$ for all $(a, b; c, d) \in \mathcal{D}$.

Interleavings in D -categories

Definition (D -categories). A D -category is a category \mathcal{C} with a strict monoidal functor \mathcal{S} from D to the category of endofunctors on \mathcal{C} . We refer to \mathcal{S} as the *smoothing functor* of \mathcal{C} .

Now let \mathcal{C} be a D -category with smoothing functor \mathcal{S} . For $a \leq 0, b \geq 0$, and an object A in \mathcal{C} we get two things, the image $\mathcal{S}((a, b))(A)$ of A under the endofunctor $\mathcal{S}((a, b))$ and a natural homomorphism from A to $\mathcal{S}((a, b))(A)$ induced by $\mathbf{o} \preceq (a, b)$. With some abuse of notation we denote this homomorphism by $\mathcal{S}((a, b))_A$. Now let A and B be objects in \mathcal{C} .

Definition (Interleavings). For $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$ an (\mathbf{a}, \mathbf{b}) -interleaving of A and B is a pair of homomorphisms $\varphi: A \rightarrow \mathcal{S}(\mathbf{a})(B)$ and $\psi: B \rightarrow \mathcal{S}(\mathbf{b})(A)$ such that

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{S}(\mathbf{a}+\mathbf{b})_A} & \mathcal{S}(\mathbf{a}+\mathbf{b})(A) \\ \varphi \searrow & & \nearrow \mathcal{S}(\mathbf{a})(\psi) \\ & \mathcal{S}(\mathbf{a})(B) & \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{\mathcal{S}(\mathbf{a}+\mathbf{b})_B} & \mathcal{S}(\mathbf{a}+\mathbf{b})(B) \\ \psi \searrow & & \nearrow \mathcal{S}(\mathbf{b})(\varphi) \\ & \mathcal{S}(\mathbf{b})(A) & \end{array}$$

commute.

We say A and B are (\mathbf{a}, \mathbf{b}) -interleaved if there is an (\mathbf{a}, \mathbf{b}) -interleaving of A and B .

Now we use the weightings on \mathcal{D} described on the previous card to describe two notions of an interleaving distance.

Definition. Let \mathcal{I} be the set of all $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$ such that there is an (\mathbf{a}, \mathbf{b}) -interleaving of A and B . Then we set

$$M_{\mathcal{S}}(A, B) := \inf(\varepsilon \circ \gamma \circ \delta)(\mathcal{I}) \quad \text{and} \quad \mu_{\mathcal{S}}(A, B) := \inf(\varepsilon \circ \delta)(\mathcal{I}).$$

We name $M(A, B)$ the *absolute interleaving distance* of A and B and $\mu(A, B)$ the *relative interleaving distance*.

Now let \mathcal{C}' be another D -category with smoothing functor \mathcal{S}' .

Definition. A 1-homomorphism of D -categories from \mathcal{C} to \mathcal{C}' is a functor G from \mathcal{C} to \mathcal{C}' such that

$$G \circ \mathcal{S}(\mathbf{a}) = \mathcal{S}'(\mathbf{a}) \circ G \quad \text{and} \quad G \circ \mathcal{S}(\mathbf{a} \preceq \mathbf{b}) = \mathcal{S}'(\mathbf{a} \preceq \mathbf{b}) \circ G$$

for all $\mathbf{a}, \mathbf{b} \in D$ with $\mathbf{a} \preceq \mathbf{b}$.

Theorem. Let G be a 1-homomorphism from \mathcal{C} to \mathcal{C}' , then

$$M_{\mathcal{S}'}(G(A), G(B)) \leq M_{\mathcal{S}}(A, B) \quad \text{and} \quad \mu_{\mathcal{S}'}(G(A), G(B)) \leq \mu_{\mathcal{S}}(A, B).$$

Persistence-enhanced Functors

Definition. Let $f: X \rightarrow \mathbb{R}$ and $g: X \rightarrow \mathbb{R}$ be continuous functions and let $(a, b), (c, d) \in D$. Then a homomorphism from $(f, (a, b))$ to $(g, (c, d))$ is a continuous map $\varphi: X \rightarrow Y$ such that $c - a \leq f(p) - g(\varphi(p)) \leq d - b$ for all $p \in X$. This defines a category which we denote by \mathcal{F} .

Next we define a smoothing functor for \mathcal{F} .

Definition. Let $f: X \rightarrow \mathbb{R}$ be a continuous function and let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in D$ with $\mathbf{b} \preceq \mathbf{c}$. Then we set $\mathcal{T}(\mathbf{b})((f, \mathbf{a})) := (f, \mathbf{a} + \mathbf{b})$ and $\mathcal{T}(\mathbf{b} \preceq \mathbf{c})_{(f, \mathbf{a})} := \text{id}_X$.

Lemma. For two \mathbb{R} -spaces f and g we have

$$M(f, g) = M_{\mathcal{T}}((f, \mathbf{o}), (g, \mathbf{o})) \quad \text{and} \quad \mu(f, g) = \mu_{\mathcal{T}}((f, \mathbf{o}), (g, \mathbf{o})).$$

Now let F be a functor from the category of \mathbb{R} -spaces to a category \mathcal{C} .

Definition. A persistence-enhancement of F is the structure of a D -category on \mathcal{C} together with a 1-homomorphism \tilde{F} from \mathcal{F} to \mathcal{C} such that $\tilde{F}((_, \mathbf{o})) = F$.

The previous lemma and the theorem from the previous card have the following

Corollary. If there is a persistence-enhancement of F with smoothing functor \mathcal{S} , then we have

$$M_{\mathcal{S}}(F(f), F(g)) \leq M(f, g) \quad \text{and} \quad \mu_{\mathcal{S}}(F(f), F(g)) \leq \mu(f, g)$$

for all \mathbb{R} -spaces f and g .

Finally we note that the *interleaving distances* provided in [Cha+09], [MBW13], [BS14], and [dMP16] can each be realized as the absolute interleaving distance of a corresponding persistence-enhanced functor.

References

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