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## Invariants of spaces over some metric space

We consider the category of locally connected topological spaces over some complete metric space  $M$ , whose objects are continuous functions to  $M$  and whose morphisms between two given functions  $f: X \rightarrow M$  and  $g: Y \rightarrow M$  are the continuous maps  $\varphi: X \rightarrow Y$  such that

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 & \searrow f & \swarrow g \\
 & & M
 \end{array}$$

commutes. In the following we will consider several invariants (mostly given as functors to other categories and mostly in the special case where  $M = \mathbb{R}$ ) under isomorphisms of objects in this category.

### Spatial Invariants

#### The Display Space

We summarize some concluding results from (Funk 1995). We assume that  $M$  is a complete metric space.

**Definition.** For a locally connected topological space  $X$ , we denote by  $\Lambda(X)$  it's set of connected components. Given an open subset  $U \subset X$  we denote by  $\Lambda(U)$  the set of connected components of  $U$ , where we augment  $U$  with the subspace topology. This defines a cosheaf on  $X$  with values in the category of sets. Given a continuous map  $f: X \rightarrow M$  we denote by  $\lambda_f$  the pushforward  $f_*\Lambda$  and obtain the functor  $\lambda$  from the category of locally connected topological spaces over  $M$  to the category of set-valued cosheaves on  $M$ .

For a set-valued (pre)cosheaf Funk (1995) provides a construction similar to the étalé space of a (pre)sheaf.

**Definition.** Given a cosheaf  $D$  on  $M$  we define its *display space*  $\text{dis } D$  as the disjoint union of costalks  $S_p := \{x \in \prod_{p \in U \in \mathcal{O}(M)} DU \mid D_{U,V}(\pi_U(x)) = \pi_V(x) \text{ for all } U \subseteq V\}$  over all  $p \in M$  where  $\mathcal{O}(M)$  is the set of open subsets of  $M$ . Further we define  $\gamma_D: \text{dis } D \rightarrow M$  by  $\gamma_D(x) = p$  for all  $x \in S_p$ . To specify a topology on  $\text{dis } D$  we provide as a basis  $\{(U, b)\}_{U \in \mathcal{O}(M), b \in DU}$  where  $(U, b) := \{x \in \gamma_D^{-1}(U) \mid \pi_U(x) = b\}$  for  $U \in \mathcal{O}(M)$  and  $b \in DU$ .

By Funk (1995 Theorem 6.1)  $\text{dis } D$  is locally connected for any cosheaf  $D$  on  $M$ .

**Assumption.** From this point on we assume all topological spaces to be locally connected.

We continue to specify a natural transformation  $\eta$  from  $\text{id}$  to  $\gamma \circ \lambda$ .

**Definition.** Given a continuous map  $f: X \rightarrow M$ ,  $x \in X$  and  $U \in \mathcal{O}(M)$  with  $f(x) \in U$  let  $[x]_U \in \lambda_f(U) = \Lambda(f^{-1}(U))$  be the connected component of  $x$ . Now we define  $\eta_f: X \rightarrow \text{dis } \lambda_f, x \mapsto ([x]_U)_{f(x) \in U \in \mathcal{O}(M)}$ .

**Lemma.** With  $\eta_f$  defined as above we have  $(\gamma \circ \lambda)_f \circ \eta_f = f$ .

*Proof.* Given  $x \in X$  we have by the definition of  $\eta_f$  that  $\eta_f(x) \in S_{f(x)}$ , hence  $(\gamma \circ \lambda)_f(\eta_f(x)) = f(x)$  by the definition of  $(\gamma \circ \lambda)_f$ .

**Lemma.** The map  $\eta_f$  as defined above is continuous.

*Proof.* Given  $U \in \mathcal{O}(M)$  and  $b \in \lambda_f(U) = \Lambda(f^{-1}(U))$  we need to show that  $\eta_f^{-1}(\{x \in (\gamma \circ \lambda)_f^{-1}(U) \mid \pi_U(x) = b\})$  is open. To do so we will show that  $\eta_f^{-1}(\{x \in (\gamma \circ \lambda)_f^{-1}(U) \mid \pi_U(x) = b\}) = b$  which is open, since  $X$  is locally connected. Suppose that  $x \in X$  is such that  $\eta_f(x) \in (\gamma \circ \lambda)_f^{-1}(U)$  and  $\pi_U(\eta_f(x)) = b$ , then  $x \in \eta_f^{-1}((\gamma \circ \lambda)_f^{-1}(U)) = f^{-1}(U)$  by the previous lemma. Further we have  $b = \pi_U(\eta_f(x)) = \pi_U([x]_U)_{f(x) \in U \in \mathcal{O}(M)} = [x]_U$ , hence  $x \in b$ . The converse follows from a similar argument.

**Definition.** Let  $f: X \rightarrow M$  be continuous, then  $f$  is a *cosheaf space over  $M$*  if  $\eta_f$  is a homeomorphism.

By (Funk 1995, Theorem 5.9 and Remark 5.10)  $\lambda$  and  $\gamma$  form a pair of adjoint functors  $\lambda \dashv \gamma$  with unit  $\eta$ . Further the counit  $\varepsilon$  for this adjunction is a natural isomorphism by (Funk 1995, Theorem 6.1). We summarize this as a

- (1) **Theorem.**  $\lambda$  and  $\gamma$  form a pair of adjoint functors  $\lambda \dashv \gamma$  with unit  $\eta$  and whose counit  $\varepsilon$  is an isomorphism.

**Corollary.** The category of cosheaves on  $M$  is equivalent to the reflective subcategory of cosheaf spaces over  $M$ .

*Proof.* This follows with (Gabriel and Zisman 1967, Proposition 1.3 or <http://ncatlab.org/nlab/show/reflective+subcategory#characterizations>).

*Remark.* Beyond the above Funk (1995 Theorem 5.17) provides a topological characterization of cosheaf spaces which hasn't been mentioned here.

## The Reeb Space

de Silva, Munch, and Patel (2015) observed that  $\gamma \circ \lambda$  is closely related to another endofunctor on topological spaces over  $M$ , the Reeb space.

**Definition.** Given a continuous map  $f: X \rightarrow M$  and  $x \in X$  let  $\pi_f(x)$  be the connected component of  $x$  in  $f^{-1}(f(x))$ . In this way we obtain a function  $\pi_f: X \rightarrow 2^X$  and we endow  $\pi_f(X)$  with the quotient topology<sup>1</sup>. By the universal property of the quotient space there is a unique continuous function  $\tilde{f}: \pi_f(X) \rightarrow M$  such that  $\tilde{f} \circ \pi_f = f$  and we define  $\rho_f = \tilde{f}$ .

With this definition  $\rho$  forms an endofunctor on topological spaces over  $M$  and  $\pi$  a natural transformation from  $\text{id}$  to  $\rho$ . Given a continuous map  $f: X \rightarrow M$  for a locally connected topological space  $X$  the universal property of the quotient space induces a unique map  $\phi_f: \pi_f(X) \rightarrow \text{dis } \lambda_f$  such that  $\phi_f \circ \pi_f = \eta_f$  and thus in particular  $\rho_f = (\gamma \circ \lambda)_f \circ \pi_f$ , hence we have the following commutative diagram

$$\begin{array}{ccc} & \text{id} & \\ \pi \swarrow & & \searrow \eta \\ \rho & \xrightarrow{\phi} & \gamma \circ \lambda \end{array}$$

in the category of endofunctors on locally connected topological spaces over  $M$ .

**Proposition.** The natural transformation  $\lambda \circ \phi$  from  $\lambda \circ \rho$  to  $\lambda \circ (\gamma \circ \lambda)$  is an isomorphism.

*Proof.* We apply  $\lambda$  to the previous diagram and obtain

$$\begin{array}{ccc} & \lambda & \\ \lambda \circ \pi \swarrow & & \searrow \lambda \circ \eta \\ \lambda \circ \rho & \xrightarrow{\lambda \circ \phi} & \lambda \circ \gamma \circ \lambda. \end{array}$$

Since  $\lambda \circ \pi$  is an isomorphism, it suffices to show that  $\lambda \circ \eta$  is an isomorphism. Given  $f: X \rightarrow M$  we apply the inverse bijection induced by the adjunction

<sup>1</sup>This is in line with the **previously made** assumption, since quotient spaces of locally connected spaces are again locally connected.

$(\lambda \dashv \gamma, \eta)$  to the diagram

$$\begin{array}{ccc}
 f & & \\
 \eta_f \downarrow & \searrow \eta_f & \\
 & & (\gamma \circ \lambda)_f \\
 & \nearrow \text{id} & \\
 (\gamma \circ \lambda)_f & & 
 \end{array}$$

and obtain

$$\begin{array}{ccc}
 \lambda_f & & \\
 (\lambda \circ \eta)_f \downarrow & \searrow \text{id} & \\
 & & \lambda_f \\
 & \nearrow (\varepsilon \circ \lambda)_f & \\
 (\lambda \circ \gamma \circ \lambda)_f & & 
 \end{array}$$

hence  $(\lambda \circ \eta)_f$  is the inverse to  $(\varepsilon \circ \lambda)_f$ .

- (2) **Corollary.**  $\phi$  and  $\eta \circ \rho$  are naturally isomorphic as functors from the category of topological spaces over  $\mathbb{R}$  to the category of homomorphisms in the category of topological spaces over  $\mathbb{R}$ .

*Example.* Let  $f: X \rightarrow \mathbb{R}$  be a [proper Morse function](#), then the critical points of  $f$  are isolated and since  $f$  is proper, it's critical values are isolated as well. Hence for each  $r \in \mathbb{R}$  there is an  $\varepsilon_r > 0$  such that for all  $0 < \delta \leq \varepsilon_r$  the inclusion of  $f^{-1}(r)$  into  $f^{-1}((r - \delta, r + \delta))$  is a homotopy equivalence and thus  $\phi_f$  is a homeomorphism.

de Silva, Munch, and Patel (2015) provide a self-contained treatment of the above when  $\lambda$  and  $\gamma$  are restricted to full subcategories of topological spaces over  $\mathbb{R}$  respectively cosheaves on  $\mathbb{R}$ . When  $\phi$  is restricted to this subcategory of topological spaces over  $\mathbb{R}$  referred to as constructible  $\mathbb{R}$ -spaces, then  $\phi$  is a natural isomorphism. Further the authors provide a geometric description of the resulting subcategory of cosheaf spaces over  $\mathbb{R}$ . They refer to this category as **Reeb** or as the category of  $\mathbb{R}$ -graphs.

### Ascending Cosheaves

In addition to the space  $\mathbb{R}$  we consider the reals augmented with a coarser topology.

**Definition.** Let  $\bar{\mathbb{R}}$  be the topological space  $(\mathbb{R}, \{(-\infty, r)\}_{-\infty \leq r \leq \infty})$ , then we have the continuous map  $\text{id}: \mathbb{R} \rightarrow \bar{\mathbb{R}}, x \mapsto x$ .

We can pushforward cosheaves on  $\mathbb{R}$  to cosheaves on  $\bar{\mathbb{R}}$  via  $\text{id}$ .

**Definition.** Given a cosheaf  $F$  on  $\mathbb{R}$  and  $-\infty \leq r \leq \infty$  we define  $\text{id}_* F((-\infty, r)) = F((-\infty, r))$ .

Similar to defining the pullback for sheaves, we take two steps to define the pullback of a cosheaf via  $\text{id}$ .

**Definition.** Given a cosheaf  $F$  on  $\bar{\mathbb{R}}$  and an open subset  $U \subseteq \mathbb{R}$  we define  $\text{id}^+ F(U) := F((-\infty, \sup U))$ .

With this definition  $\text{id}^+ F$  is merely a precosheaf for all we know. Yet we have the following.

**Lemma.** Given a cosheaf  $F$  on  $\bar{\mathbb{R}}$  the precosheaf  $\text{id}^+ F$  is a cosheaf on the poset of open intervals.

*Proof.* Let  $(a, b) = \bigcup_{i \in I} (a_i, b_i)$ . Without loss of generality we assume that  $I$  has a linear order such that  $b_i \leq b_j$  for all  $i \leq j$  and  $a_i \leq a_j$  for all  $i \leq j$  with  $b_i = b_j$ . Since  $b = \sup_{i \in I} \sup(a_i, b_i) = \sup_{i \in I} b_i$  and since  $F$  is a cosheaf we have the coequalizer diagram

$$\coprod_{i < j} F((-\infty, b_i)) \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\sigma'} \end{array} \coprod_{i \in I} F((-\infty, b_i)) \longrightarrow F((-\infty, b))$$

where  $\sigma_{i,j}$  maps  $F((-\infty, b_i))$  identical to  $F((-\infty, b_i))$  and  $\sigma'_{i,j}$  maps  $F((-\infty, b_i))$  to  $F((-\infty, b_j))$  via the induced inclusion. Now suppose we have  $i < j$  such that  $b_i \leq a_j$ , then since  $[b_i, a_j]$  is compact we can find  $i < i_1 < \dots < i_k$  such that  $[b_i, a_j] \subseteq \bigcup_{l=1}^k (a_{i_l}, b_{i_l})$  and such that this cover is minimal. If  $i_k < j$  we set  $\tau = \sigma_{i_k, j}$ ,  $\tau' = \sigma'_{i_k, j}$  and if  $j < i_k$  we set  $\tau = \sigma_{j, i_k}$ ,  $\tau' = \sigma'_{j, i_k}$ . With this we have  $\sigma'_{i,j} \circ \sigma_{i,j}^{-1} = \tau' \circ \tau^{-1} \circ \sigma'_{i_{k-1}, i_k} \circ \sigma_{i_{k-1}, i_k}^{-1} \circ \dots \circ \sigma'_{i_1, i_2} \circ \sigma_{i_1, i_2}^{-1} \circ \sigma'_{i, i_1} \circ \sigma_{i, i_1}^{-1}$  and yet at the same time  $b_j > a_{i_k}$ ,  $b_{i_k} > a_j$ ,  $b_{i_{k-1}} > a_{i_k}$ ,  $\dots$ ,  $b_{i_1} > a_{i_2}$ , and  $b_i > a_{i_1}$ , since  $[b_i, a_j]$  is connected and the cover is minimal. Thus we may omit all terms from the leftmost coproduct in the above diagram where  $b_i \leq a_j$  without losing the property of it being a coequalizer diagram. Now for any  $i < j$  such that  $b_i > a_j$  we have  $\text{id}^+ F((a_i, b_i) \cap (a_j, b_j)) = F((-\infty, b_i))$ , hence we may replace the corresponding term in the leftmost coproduct by  $\text{id}^+ F((a_i, b_i) \cap (a_j, b_j))$ . Similarly we may replace the terms in the middle and the term on the right to arrive at a coequalizer diagram of the form

$$\coprod_{i < j, b_i > a_j} \text{id}^+ F((a_i, b_i) \cap (a_j, b_j)) \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\sigma'} \end{array} \coprod_{i \in I} \text{id}^+ F((a_i, b_i)) \longrightarrow \text{id}^+ F((a, b)).$$

Now for  $i < j$  such that  $b_i \leq a_j$  we have  $\text{id}^+ F((a_i, b_i) \cap (a_j, b_j)) = \text{id}^+ F(\emptyset) = F(\emptyset) = \emptyset$  which does not contribute to the coequalizer and this implies the claim.

**Definition.** Given a cosheaf  $F$  on  $\bar{\mathbb{R}}$  and an open subset  $U \subseteq \mathbb{R}$  we define  $\text{id}^{-1} F(U) := \varinjlim_{(a,b) \subseteq U} \text{id}^+ F((a, b))$ .

By the previous lemma  $\text{id}^{-1} F$  as defined above is a cosheaf.

*Remark.*  $\text{id}^{-1} F$  as defined above is isomorphic to the cosheafification of  $\text{id}^+ F$  or the cosheaf associated to  $\text{id}^+ F$ , see for example (Funk 1995, Theorem 6.3 and Remark 6.4).

**Definition.** Given a cosheaf  $F$  on  $\mathbb{R}$  we define a homomorphism  $\eta'_F$  of cosheaves from  $F$  to  $\text{id}^{-1} \text{id}_* F$ . Since both are cosheaves it suffices to define  $\eta'_F$  on open intervals. So for  $-\infty \leq a < b \leq \infty$  we define  $\eta'_F$  from  $F((a, b))$  to  $\text{id}^{-1} \text{id}_* F((a, b)) = \text{id}^+ \text{id}_* F((a, b)) = F((-\infty, b))$  to be the map induced by the inclusion  $(a, b) \subseteq (-\infty, b)$ .

**Definition.** Let  $F$  be a cosheaf on  $\mathbb{R}$ , then  $F$  is *ascending* if  $\eta'_F$  is an isomorphism.

- (3) **Proposition.**  $\text{id}_*$  and  $\text{id}^{-1}$  form a pair of adjoint functors  $\text{id}_* \dashv \text{id}^{-1}$  with unit  $\eta'$  and whose counit  $\varepsilon'$  is an isomorphism.

*Proof.* Let  $F$  be a cosheaf on  $\mathbb{R}$ , let  $G$  be a cosheaf on  $\bar{\mathbb{R}}$ , and let  $g$  be a homomorphism from  $F$  to  $\text{id}^{-1} G$ . Now suppose we have a morphism  $f$  from  $\text{id}_* F$  to  $G$  such that  $(\text{id}^{-1} f) \circ \eta'_F = g$ , then for any  $r \in \mathbb{R}$  we have  $g_{(-\infty, r)} = ((\text{id}^{-1} f) \circ \eta'_G)_{(-\infty, r)} = f_{(-\infty, r)}$  and this determines  $f$ . Now suppose  $f$  is defined by  $g_{(-\infty, r)} = f_{(-\infty, r)}$  for any  $r \in \mathbb{R}$  and we have  $-\infty \leq a < b \leq \infty$ , then  $g_{(a, b)}$  is the same as  $g_{(-\infty, b)}$  pre-composed with the map induced by inclusion from  $F((a, b))$  to  $F((-\infty, b))$  by naturality. But this is the same as  $((\text{id}^{-1} f) \circ \eta'_G)_{(a, b)}$  by definition of  $f$ , hence  $g$  and  $(\text{id}^{-1} f) \circ \eta'_G$  agree on a basis of  $\mathbb{R}$ .

By the above argument  $\varepsilon'_G$  is equal to  $\text{id}_{\text{id}^{-1} G}$  when restricted to  $\text{id}_* \text{id}^{-1} G((-\infty, r)) = \text{id}^{-1} G((-\infty, r)) = G((-\infty, r))$ , hence  $\varepsilon'_G$  is an isomorphism.

**Corollary.** The category of cosheaves on  $\bar{\mathbb{R}}$  is equivalent to the reflective subcategory of ascending cosheaves on  $\mathbb{R}$ .

*Proof.* This follows with (Gabriel and Zisman 1967, Proposition 1.3 or <http://ncatlab.org/nlab/show/reflective+subcategory#characterizations>).

## Ascending Spaces

Later we will make the ascending cosheaf  $\text{id}^{-1} \text{id}_* \lambda_f$  for a continuous function  $f$  the cosheaf version of the join tree associated to  $f$ . As an intermediate step we show that we can obtain this cosheaf not only by post-composing  $\lambda$  with  $\text{id}^{-1} \text{id}_*$  but also by pre-composing  $\lambda$  with another functor, the epigraph. This use of the epigraph in defining the join tree<sup>2</sup> is due to Morozov, Beketayev, and Weber (2013).

**Definition.** Let  $f: X \rightarrow \mathbb{R}$  be a continuous map, its *epigraph* is  $\text{epi } f := \{(x, y) \in X \times \mathbb{R} \mid y \geq f(x)\}$ . Further we define  $\iota_f: \text{epi } f \rightarrow \mathbb{R}, (x, y) \mapsto y$  and  $\kappa_f: X \rightarrow \text{epi } f, x \mapsto (x, f(x))$ .

<sup>2</sup>Though join trees are referred to as merge trees in the cited paper.

With these definitions  $\iota$  defines a functor on topological spaces over  $\mathbb{R}$  with  $\kappa$  a natural transformation from  $\text{id}$  to  $\iota$ .

- (4) **Definition.** A function  $f: X \rightarrow \mathbb{R}$  is *ascending* if for all  $r \in \mathbb{R}$  there is a continuous map  $H_r: X \times [0, 1] \rightarrow X$  such that  $H_r(x, t) = x$  for all  $0 \leq t \leq 1$  and  $x \in X$  with  $f(x) \geq r$  and such that  $f(H_r(x, t)) = r + t(f(x) - r)$  for all  $0 \leq t \leq 1$  and  $x \in X$  with  $f(x) \leq r$ .
- (5) **Lemma.** For any continuous function  $f: X \rightarrow \mathbb{R}$  the projection  $\iota_f: \text{epi } f \rightarrow \mathbb{R}$  is ascending.

*Proof.* For  $r \in \mathbb{R}$  we set  $H_r: \text{epi } f \times [0, 1] \rightarrow \text{epi } f, ((x, y), t) \mapsto (x, \max\{r + t(y - r), y\})$ .

- (6) **Lemma.** For any ascending function  $f: X \rightarrow \mathbb{R}$  the cosheaf  $\lambda_f$  is ascending as well.

*Proof.* Given  $-\infty \leq a < r < b \leq \infty$  we proof that the maps from  $\Lambda(f^{-1}([r, b]))$  to  $\Lambda(f^{-1}((a, b))) = \lambda_f((a, b))$  respectively  $\Lambda(f^{-1}((-\infty, b))) = \lambda_f((-\infty, b))$  induced by the inclusions are bijections<sup>3</sup>. From this our claim follows. Since inclusions as maps of spaces always commute the two bijections commute with  $(\eta' \circ \lambda)_f$  as well, hence  $(\eta' \circ \lambda)_f$  is a bijection as a map from  $\lambda_f((a, b))$  to  $\lambda_f((-\infty, b)) = \text{id}_* \lambda_f((-\infty, b)) = \text{id}^+ \text{id}_* \lambda_f((a, b)) = \text{id}^{-1} \text{id}_* \lambda_f((a, b))$ . And since the open intervals of  $\mathbb{R}$  form a basis, the lemma follows.

Given any point  $x \in f^{-1}((a, r))$  the map  $t \mapsto H_r(x, t)$  defines a continuous path in  $f^{-1}((a, b))$  from  $H_r(x, 0) \in f^{-1}([r, b])$  to  $x$ , hence induced map from  $\Lambda(f^{-1}([r, b]))$  to  $\Lambda(f^{-1}((a, b))) = \lambda_f((a, b))$  is surjective. Now suppose  $x, y \in f^{-1}([r, b])$  lie in the same connected component  $C$  of  $f^{-1}((a, b))$ , then  $H_r(C, 0)$  is connected since  $H_r$  is continuous. Further  $x, y \in H_r(C, 0)$ , hence the induced map from  $\Lambda(f^{-1}([r, b]))$  to  $\lambda_f((a, b))$  is injective. The induced map from  $\Lambda(f^{-1}([r, b]))$  to  $\lambda_f((-\infty, b))$  is a bijection by a similar argument.

*Remark.* The previous result remains valid if instead of  $\lambda$  we consider the pushforward of another cosheaf on  $X$  that maps inclusions of open sets in  $X$  that are homotopy equivalences to bijections of sets.

- (7) **Lemma.** Given a continuous map  $f: X \rightarrow \mathbb{R}$  the homomorphism  $\text{id}_*(\lambda \circ \kappa)_f$  from  $\text{id}_* \lambda_f$  to  $\text{id}_*(\lambda \circ \iota)_f$  is an isomorphism.

*Proof.* Given  $b \in \mathbb{R} \cup \{\infty\}$  we show that  $\kappa_f(f^{-1}((-\infty, b))) = \{(x, f(x))\}_{\{x \in X \mid f(x) < b\}}$  is a strong deformation retract of  $\iota_f^{-1}((-\infty, b)) = \{(x, y) \in X \times (-\infty, b) \mid y \geq f(x)\}$ . Then the result follows by a similar argument as the previous lemma. We define  $R: \iota_f^{-1}((-\infty, b)) \times [0, 1] \rightarrow \iota_f^{-1}((-\infty, b)), ((x, y), t) \mapsto (x, f(x) + t(y - f(x)))$ , then  $R((x, y), 1) = (x, y)$  and  $R((x, y), 0) = (x, f(x))$  for all  $(x, y) \in \iota_f^{-1}((-\infty, b))$ .

<sup>3</sup>The space  $f^{-1}([r, b])$  may not be locally connected. However we won't need this property.

- (8) **Proposition.** The natural transformations  $\eta' \circ \lambda$  and  $\lambda \circ \kappa$  are isomorphic as objects in the category of functors from topological spaces over  $\mathbb{R}$  to cosheaves on  $\mathbb{R}$  under  $\lambda$ . In particular  $\text{id}^{-1} \text{id}_* \lambda$  and  $\lambda \circ \iota$  are naturally isomorphic.

*Proof.* Given  $f: X \rightarrow \mathbb{R}$  we have the commutative diagram

$$\begin{array}{ccc} \lambda_f & \xrightarrow{(\lambda \circ \kappa)_f} & (\lambda \circ \iota)_f \\ \downarrow (\eta' \circ \lambda)_f & & \downarrow (\eta' \circ \lambda \circ \iota)_f \\ \text{id}^{-1} \text{id}_* \lambda_f & \xrightarrow{\text{id}^{-1} \text{id}_* (\lambda \circ \kappa)_f} & \text{id}^{-1} \text{id}_* (\lambda \circ \iota)_f. \end{array}$$

By lemma 5 and lemma 6 the homomorphism  $(\eta' \circ \lambda \circ \iota)_f$  is an isomorphism. And by lemma 7 we have that  $\text{id}^{-1} \text{id}_* (\lambda \circ \kappa)_f$  is an isomorphism.

## The Join Tree

The following definition<sup>4</sup> is from (Morozov, Beketayev, and Weber 2013).

**Definition.** Let  $f: X \rightarrow \mathbb{R}$  be a continuous map we define its *join tree* to be the continuous map  $(\rho \circ \iota)_f$  from  $(\pi \circ \iota)_f(\text{epi } f)$  to  $\mathbb{R}$ .

With this definition  $\rho \circ \iota$  is an endofunctor on topological spaces over  $\mathbb{R}$ . Given a continuous map  $f: X \rightarrow \mathbb{R}$  we have  $(\pi \circ \iota)_f \circ \kappa_f = (\rho \circ \kappa)_f \circ \pi_f$ , so in somewhat sloppy notation  $(\pi \circ \iota) \circ \kappa = (\rho \circ \kappa) \circ \pi$  is a natural transformation from  $\text{id}$  to  $\rho \circ \iota$ . Similarly we have the function  $(\gamma \circ \lambda \circ \iota)_f$  defined on the display space  $\text{dis}(\lambda \circ \iota)_f$  of  $(\lambda \circ \iota)_f$ . And just as with  $\rho$  we have the natural transformation  $(\eta \circ \iota) \circ \kappa = (\gamma \circ \lambda \circ \kappa) \circ \eta$  from  $\text{id}$  to  $\gamma \circ \lambda \circ \iota$ . The two constructions are related via the commutative diagram

$$\begin{array}{ccccc} & & \rho_f & \xrightarrow{(\rho \circ \kappa)_f} & (\rho \circ \iota)_f \\ & \nearrow \pi_f & \downarrow \phi_f & & \downarrow (\phi \circ \iota)_f \\ f & & (\gamma \circ \lambda)_f & \xrightarrow{(\gamma \circ \lambda \circ \kappa)_f} & (\gamma \circ \lambda \circ \iota)_f \\ & \searrow \eta_f & & & \end{array}$$

given a function  $f: X \rightarrow \mathbb{R}$ . In the [section on the Reeb space](#) we considered the left triangle which suggests to replace the Reeb graph functor  $\rho$  and the natural transformation  $\pi$  by  $\gamma \circ \lambda$  and  $\eta$  respectively. Now  $\rho \circ \kappa$  yields a nice and classic map from any Reeb graph to the corresponding join tree, so our replacement of the Reeb graph functor  $\rho$  by  $\gamma \circ \lambda$  is only complete, if also we can replace the join tree functor  $\rho \circ \iota$  and the natural transformation  $\rho \circ \kappa$  and if we can extend  $\phi$  to a natural transformation from  $\rho \circ \kappa$  to its replacement. And here the commutative square on the right hand side, suggests we may take  $\lambda \circ \gamma \circ \iota$  as a replacement for the join tree functor  $\rho \circ \iota$  and to take  $\gamma \circ \lambda \circ \kappa$  as a replacement for  $\rho \circ \kappa$ , since then we can extend  $\phi$  by  $\phi \circ \iota$  to a natural transformation from  $\rho \circ \kappa$  to  $\gamma \circ \lambda \circ \kappa$ . We further note that by

<sup>4</sup>Though join trees are referred to as merge trees in the cited paper.



corollary 2 in that section the natural transformation  $(\phi, \phi \circ \iota)$  from  $\rho \circ \kappa$  to  $\gamma \circ \lambda \circ \kappa$  is isomorphic to the natural transformation  $(\eta \circ \rho, \eta \circ \rho \circ \iota)$  from  $\rho \circ \kappa$  to  $\gamma \circ \lambda \circ \rho \circ \kappa$ , so our choice of replacements is the same as if we applied  $\gamma \circ \lambda$  to the upper row in the diagram. And by proposition 8 we have a natural isomorphism from  $(\gamma \circ \lambda \circ \iota)$  to  $\gamma \text{id}^{-1} \text{id}_* \lambda$  that commutes with  $\gamma \circ \lambda \circ \kappa$  and  $\gamma \circ \eta' \circ \lambda$  so that we can use the following

**Proposition.**  $\text{id}_* \lambda$  and  $\gamma \text{id}^{-1}$  form a pair of adjoint functors  $\text{id}_* \lambda \dashv \gamma \text{id}^{-1}$  with unit  $(\gamma \circ \eta' \circ \lambda) \circ \eta$  and whose counit is an isomorphism.

*Proof.* The first statement follows from theorem 1, proposition 3 and the general statement that the two pairs of adjoint functors, when composed in the same way as in our claim, form again a pair of adjoint functors with the unit described as in the claim, see for example [https://en.wikipedia.org/wiki/Adjoint\\_functors#Composition](https://en.wikipedia.org/wiki/Adjoint_functors#Composition). And for the counit of this composed adjunction we have the formula  $\text{id}_* \varepsilon \text{id}^{-1} \circ \varepsilon'$ . By theorem 1  $\varepsilon$  is an isomorphism, hence  $\text{id}_* \varepsilon \text{id}^{-1}$  is an isomorphism and by proposition 3  $\varepsilon'$  is an isomorphism and thus our claim follows.

**Corollary.** The category of cosheaves on  $\bar{\mathbb{R}}$  is equivalent to the reflective subcategory of ascending cosheaf spaces over  $\mathbb{R}$ .

*Proof.* By Gabriel and Zisman (1967 Proposition 1.3 or <http://ncatlab.org/nlab/show/reflective+subcategory#characterizations>) the category of cosheaves on  $\bar{\mathbb{R}}$  is equivalent to the reflective subcategory of those spaces  $f: X \rightarrow \mathbb{R}$  over  $\mathbb{R}$  for which  $(\gamma \circ \eta' \circ \lambda)_f \circ \eta_f$  is an isomorphism. Now suppose this is the case for  $f$ , then  $f$  is isomorphic to  $\gamma \text{id}^{-1} \text{id}_* \lambda$  which is in the image of  $\gamma$  and thus a cosheaf space, hence  $\eta_f$  is an isomorphism. From this it follows that  $(\gamma \circ \eta' \circ \lambda)_f$  is an isomorphism as well, hence by proposition 8  $(\gamma \circ \lambda \circ \kappa)_f$  is an isomorphism. Now we consider the commutative diagram

$$\begin{array}{ccc} f & \xrightarrow{\kappa_f} & \iota_f \\ \eta_f \downarrow & & \downarrow (\eta \circ \iota)_f \\ (\gamma \circ \lambda)_f & \xrightarrow{(\gamma \circ \lambda \circ \kappa)_f} & (\gamma \circ \lambda \circ \iota)_f. \end{array}$$

Hence we have the retract<sup>5</sup>  $R := \eta_f^{-1} \circ (\gamma \circ \lambda \circ \kappa)_f^{-1} \circ (\eta \circ \iota)_f$  from  $\iota_f$  to  $f$ . By lemma 5  $\iota_f$  is ascending, so given  $r \in \mathbb{R}$  there is a map  $H_r$  as in definition 4. Now let  $\tilde{H}_r: X \times [0, 1] \rightarrow X$  be defined by  $\tilde{H}_r(x, t) = r(H_r(\kappa_f(x), t))$  then  $\tilde{H}_r$  inherits the properties needed in order for  $f$  to be ascending. Conversely if  $f$  is an ascending cosheaf space over  $\mathbb{R}$ , then  $\eta_f$  is an isomorphism since  $f$  is a cosheaf space. And by lemma 6  $\lambda_f$  is ascending, hence  $(\eta' \circ \lambda)_f$  is an isomorphism.

In conclusion  $(\gamma \circ \lambda \circ \iota)_f$  is an ascending cosheaf space over  $\mathbb{R}$  given a function  $f$ . It's cosheaf of connected components  $(\lambda \circ \gamma \circ \lambda \circ \iota)_f$  is isomorphic to  $(\lambda \circ \iota)_f$  by theorem 1. By lemma 5 and lemma 6  $(\lambda \circ \iota)_f$  is ascending, and thus we have an associated cosheaf  $\text{id}_*(\lambda \circ \iota)_f$  on  $\bar{\mathbb{R}}$  via the adjunction  $\text{id}_* \dashv \text{id}^{-1}$  by proposition 3. By lemma 7 this

<sup>5</sup>By a retract we mean a homomorphism  $R$  in the category of topological spaces over  $\mathbb{R}$  from  $\iota_f$  to  $f$  such that  $R \circ \kappa_f = \text{id}$ .

cosheaf is isomorphic to  $\text{id}_* \lambda_f$  which is the cosheaf on  $\bar{\mathbb{R}}$  associated to  $\gamma \text{id}^{-1} \text{id}_* \lambda_f$  via the adjunction  $\text{id}_* \lambda \dashv \gamma \text{id}^{-1}$ . Now applying  $\text{id}^{-1}$  to  $\text{id}_* \lambda_f \cong \text{id}_*(\lambda \circ \iota)_f$  recovers  $(\lambda \circ \iota)_f$ , hence  $(\gamma \circ \lambda \circ \iota)_f$  and  $\gamma \text{id}^{-1} \text{id}_* \lambda_f$  are isomorphic and thus a posteriori  $\text{id}_* \lambda_f$  is the cosheaf on  $\bar{\mathbb{R}}$  associated to the ascending cosheaf space  $(\gamma \circ \lambda \circ \iota)_f$  via the adjunction  $\text{id}_* \lambda \dashv \gamma \text{id}^{-1}$ . (Here the author allowed himself some redundancy repeating the proof of proposition 8.)

## From Cosheaves to Sheaves

### From Sets to Algebras

For an integral domain  $A$  we consider the contravariant functor  $\text{hom}(\_, A)$  from the category of sets to the category of commutative unital  $A$ -algebras. We note that since  $A$  is an integral domain the idempotents of  $\text{hom}(L, A)$  for any set  $L$  are precisely the maps from  $L$  to  $A$  with values in  $\{0, 1\}$ .

(9) **Lemma.**  $\text{hom}(\_, A)$  is [pseudomononic](#).

*Proof.*  $\text{hom}(\_, A)$  is faithful since for any map  $m: L \rightarrow K$  and  $k \in K$  we have  $m^{-1}(k) = (\text{hom}(m, A)(1_k))^{-1}(1)$  where  $1_k := 1_{\{k\}}$  and  $1_{K'}$  is the [indicator function](#) for any subset  $K' \subseteq K$ .

Now suppose  $\varphi$  is an isomorphism from  $\text{hom}(K, A)$  to  $\text{hom}(L, A)$  then  $\varphi$  induces a bijection between the non-zero [centrally primitive](#) idempotents of  $\text{hom}(K, A)$  and  $\text{hom}(L, A)$ . Now the non-zero centrally primitive idempotents of  $\text{hom}(K, A)$  are just the maps of the form  $1_k$  for some  $k \in K$  and similarly for  $\text{hom}(L, A)$ . Let  $m: L \rightarrow K$  be the corresponding inverse bijection, then for any  $c \in \text{hom}(K, A)$  and  $l \in L$  we have

$$\begin{aligned} \varphi(c) \cdot 1_l &= \varphi(c \cdot 1_{m(l)}) = \varphi(c(m(l))1_{m(l)}) \\ &= c(m(l))\varphi(1_{m(l)}) = c(m(l))1_l \\ &= \text{hom}(m, A)(c) \cdot 1_l \end{aligned}$$

and thus  $\varphi = \text{hom}(m, A)$ .

**Corollary.** The functor  $\text{hom}(\_, A)$  induces an anti-equivalence between the category of sets and the replete image of  $\text{hom}(\_, A)$ .

**Corollary.** For any category  $\mathcal{C}$  the functor  $\text{hom}(\_, A)$  induces an anti-equivalence between the category of set-valued precosheaves on  $\mathcal{C}$  and the category of presheaves with values in the replete image of  $\text{hom}(\_, A)$ .

**Lemma.**  $\text{hom}(\_, A)$  is full when restricted to the category of finite sets.

*Proof.* Let  $\varphi: \text{hom}(K, A) \rightarrow \text{hom}(L, A)$  be a homomorphism with  $K$  and  $L$  finite, then  $\varphi(1_k)$  is an idempotent for each  $k \in K$  and thus we have subsets  $L_k \subseteq L$  such that  $\varphi(1_k) = 1_{L_k}$ . Further we have  $\sum_{l \in L} 1_l = 1 = \varphi(1) = \varphi(\sum_{k \in K} 1_k) = \sum_{k \in K} \varphi(1_k) = \sum_{k \in K} 1_{L_k}$  and thus  $L = \bigcup_{k \in K} L_k$ . Now for any  $k, k' \in K$  with  $k \neq k'$  we have  $0 = \varphi(0) = \varphi(1_k \cdot 1_{k'}) = 1_{L_k} \cdot 1_{L'_k}$ , hence  $L_k$  and  $L_{k'}$  are disjoint. Altogether we obtain that the subsets  $L_k$  with  $k \in K$  form a partition of  $L$  and

we may define a map  $m: L \rightarrow K$  such that  $m(l) = k$  for  $l \in L_k$  for all  $k \in K$ . With this definition we have  $\varphi = \text{hom}(m, A)$  since the two maps agree on a basis of  $\text{hom}(K, A)$ .

The following example shows that we cannot assume the unrestricted functor  $\text{hom}(\_, A)$  to be full, if  $A$  is a general ring.

*Example.* We consider  $\text{hom}(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})$ . Let  $\mathfrak{a}$  be the ideal of all  $c \in \text{hom}(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})$  with  $c^{-1}(0)$  cofinite. By [Krull's theorem](#)  $\text{hom}(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})$  has a maximal ideal  $\mathfrak{m}$  with  $\mathfrak{a} \subset \mathfrak{m}$  and this gives a homomorphism of fields  $i: \mathbb{Z}/p\mathbb{Z} \rightarrow \text{hom}(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})/\mathfrak{m}$ . We further have  $[c]^p - [c] = [c^p - c] = 0$  for all  $[c] \in \text{hom}(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})/\mathfrak{m}$  and as  $X^p - X$  is a polynomial of degree  $p$  it has at most  $p$  roots in  $\text{hom}(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})/\mathfrak{m}$  and thus  $i$  is a bijection. Now the canonical homomorphism from  $\text{hom}(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})$  to the quotient  $\text{hom}(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})/\mathfrak{m}$  yields a homomorphism  $\varphi: \text{hom}(\mathbb{N}, \mathbb{Z}/p\mathbb{Z}) \rightarrow \text{hom}(\{1\}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$  which is not in the image of  $\text{hom}(\_, \mathbb{Z}/p\mathbb{Z})$ , since for any map  $m: \{1\} \rightarrow \mathbb{N}$  the element  $1_{m(1)} \in \mathfrak{a} \subset \mathfrak{m}$  is mapped to  $1 \in \mathbb{Z}/p\mathbb{Z}$  under  $\text{hom}(m, \mathbb{Z}/p\mathbb{Z})$ .

*Remark.* From a discussion similar to that of the previous lemma and example we can conclude that for sets  $K$  and  $L$  with  $L$  non-empty, the map from  $\text{hom}(L, K)$  to  $\text{hom}_{A\text{-algebras}}(\text{hom}(K, A), \text{hom}(L, A))$  induced by  $\text{hom}(\_, A)$  is surjective if and only if all ideals<sup>6</sup>  $\mathfrak{p}$  of  $\text{hom}(K, A)$ , with  $\text{hom}(K, A)/\mathfrak{p} \cong A$  as  $A$ -algebras, are of the form  $\{c \in \text{hom}(K, A) \mid c(k) = 0\}$  for some  $k \in K$ .

**Lemma.**  $\text{hom}(\_, A)$  is continuous as a functor from the opposed category of sets to the category of  $A$ -algebras.

*Proof.* We argue that  $\text{hom}(\_, A)$  is continuous as a functor to the category of commutative rings, the lemma then follows by [a general result about limits in the under category](#). We fix a small category  $D$ . For an object  $X$  of any category  $\mathcal{C}$  we denote by  $\Delta(X)$  the constant functor from  $D$  to  $\mathcal{C}$  that maps any object of  $D$  to  $X$  and any morphism of  $D$  to the identity. Let  $F$  be a functor from  $D$  to the category of sets, then we have the canonical natural transformation  $t: F \rightarrow \Delta(\text{colim}(F))$ . Now  $\text{hom}(\Delta(\text{colim}(F)), A) = \Delta(\text{hom}(\text{colim}(F), A))$  and by the universal property of the limit of  $\text{hom}(F(\_), A)$  we have a homomorphism of rings  $s: \lim(\text{hom}(F(\_), A)) \rightarrow \text{hom}(\text{colim}(F), A)$  such that  $(\text{hom}(\_, A) \circ t) \circ \Delta(s)$  is the canonical natural transformation from  $\Delta(\lim(\text{hom}(F(\_), A)))$  to  $\text{hom}(F(\_), A)$ . Now the forgetful functor from the category of commutative rings to the category of sets is continuous as well as  $\text{hom}(\_, A)$  as a functor to the category of sets, hence in the category of sets both  $(\text{hom}(\_, A) \circ t) \circ \Delta(s)$  and  $\text{hom}(\_, A) \circ t$  itself satisfy the universal property of the limit of  $\text{hom}(F(\_), A)$ , and thus  $s$  is a bijection.

**Corollary.** If  $D$  is a set-valued cosheaf, then  $\text{hom}(D(\_), A)$  defines a sheaf with values in the category of  $A$ -algebras.

*Example.* For any locally path connected topological space  $X$  the singular homology  $H_0(X)$  is naturally isomorphic to the free abelian group with basis  $\Lambda(X)$  and by the universal property of the free abelian group the restriction from

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<sup>6</sup>which are prime necessarily

$\text{hom}_{\mathbb{Z}}(H_0(X), A)$  to  $\text{hom}(\Lambda(X), A)$  is an isomorphism of  $A$ -modules. Further we have a natural isomorphism of  $A$ -modules from  $H^0(X, A)$  to  $\text{hom}_{\mathbb{Z}}(H_0(X), A)$  by the universal coefficient theorem and since for any  $x \in X$  and  $\alpha, \beta \in H^0(X, A)$  we have

$$\begin{aligned} \langle \alpha \cup \beta, [x] \rangle &= \langle H^0(d, A)(\alpha \times \beta), [x] \rangle = \langle \alpha \times \beta, H_0(d)([x]) \rangle \\ &= \langle \alpha \times \beta, [(x, x)] \rangle = \langle \alpha \times \beta, [x] \times [x] \rangle \\ &= \langle \alpha, [x] \rangle \langle \beta, [x] \rangle, \end{aligned}$$

where  $d: X \rightarrow X \times X, x \mapsto (x, x)$  is the diagonal map, the composition of these two isomorphisms is an isomorphism of  $A$ -algebras. Since the above identifications are natural in  $X$ , the functors  $\text{hom}(\Lambda(\_), A)$  and  $H^0(\_, A)$  define isomorphic sheaves on any locally path connected topological space.

Given a continuous function  $f: X \rightarrow M$  from a locally path connected topological space  $X$  to  $M$ , the sheaves  $f_* \text{hom}(\Lambda(\_), A) \cong f_* H^0(\_, A)$  and  $\text{hom}(\lambda_f(\_), A)$  are identical. Bubenik, de Silva, and Scott (2014) define a generalized persistence module on the poset of open sets of  $M$  to be a functor to another category, thus  $\lambda_f$  is a generalized persistence module with values in the opposed category of sets and  $f_* H^0(\_, A)$  is a persistence module with values in the category of  $A$ -algebras. A functor from a category  $\mathcal{C}$  to a category  $\mathcal{D}$  then gives rise to a map from the generalized persistence modules with values in  $\mathcal{C}$  to persistence modules with values in  $\mathcal{D}$ , so in their language  $f_* H^0(\_, A)$  is the image of  $\lambda_f$  under the map induced by  $\text{hom}(\_, A)$  and thus their theory can be used to relate these two constructions in the context of topological persistence.

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