Contents

Invariants of spaces over some metric space

We consider the category of locally connected topological spaces over some complete metric space M, whose objects are continuous functions to M and whose morphisms between two given functions $f\colon X \to M$ and $g\colon Y \to M$ are the continuous maps $\varphi\colon X \to Y$ such that

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & M & \end{tikzcd}$$

commutes. In the following we will consider several invariants (mostly given as functors to other categories and mostly in the special case where $M = \mathbb{R}$) under isomorphisms of objects in this category.

Spatial Invariants

The Display Space

We summarize some concluding results from (Funk 1995). We assume that M is a complete metric space.

> **Definition.** For a locally connected topological space X, we denote by $\Lambda(X)$ it's set of connected components. Given an open subset $U \subset X$ we denote by $\Lambda(U)$ the set of connected components of U, where we augment U with the subspace topology. This defines a cosheaf on X with values in the category of sets. Given a continuous map $f\colon X \to M$ we denote by λ_f the pushforward $f_* \Lambda$ and obtain the functor λ from the category of locally connected topological spaces over M to the category of set-valued cosheaves on M.

For a set-valued (pre)cosheaf Funk (1995) provides a construction similar to the étalé space of a (pre)sheaf.

Definition. Given a cosheaf D on M we define it's *display space* $\operatorname{dis} D$ as the disjoint union of costalks $S_p := \big\{x \in \prod_{p\in U\in\mathcal{O}(M)} DU \;\big|\; D_{U,V}\big(\pi_U(x)\big) = \pi_V(x) \;\text{ for all }\; U \subseteq V\big\}$ over all $p \in M$ where $\mathcal{O}(M)$ is the set of open subsets of M. Further we define $\gamma_D\colon \operatorname{dis} D \to M$ by $\gamma_D(x) = p$ for all $x \in S_p$. To specify a topology on $\operatorname{dis} D$ we provide as a basis $\{(U,b)\}_{U\in\mathcal{O}(M),\; b\in DU}$ where $(U,b) := \{x \in \gamma_D^{-1}(U) \mid \pi_U(x) = b\}$ for $U \in \mathcal{O}(M)$ and $b \in DU$.

By Funk (1995 Theorem 6.1) $\operatorname{dis} D$ is locally connected for any cosheaf D on M.

Assumption. From this point on we assume all topological spaces to be locally connected.

We continue to specify a natural transformation η from id to $\gamma \circ \lambda$.

Definition. Given a continuous map $f\colon X \to M$, $x \in X$ and $U \in \mathcal{O}(M)$ with $f(x) \in U$ let $[x]_U \in \lambda_f(U) = \Lambda(f^{-1}(U))$ be the connected component of x. Now we define $\eta_f\colon X \to \operatorname{dis}\lambda_f, x \mapsto ([x]_U)_{f(x)\in U\in\mathcal{O}(M)}$.

Lemma. With η_f defined as above we have $(\gamma\circ\lambda)_f \circ \eta_f = f$.

Proof. Given $x \in X$ we have by the definition of η_f that $\eta_f(x) \in S_{f(x)}$, hence $(\gamma\circ\lambda)_f(\eta_f(x)) = f(x)$ by the definition of $(\gamma\circ\lambda)_f$.

Lemma. The map η_f as defined above is continuous.

Proof. Given $U \in \mathcal{O}(M)$ and $b \in \lambda_f(U) = \Lambda(f^{-1}(U))$ we need to show that $\eta_f^{-1}(\{x \in (\gamma\circ\lambda)_f^{-1}(U) \mid \pi_U(x) = b\})$ is open. To do so we will show that $\eta_f^{-1}(\{x \in (\gamma\circ\lambda)_f^{-1}(U) \mid \pi_U(x) = b\}) = b$ which is open, since X is locally connected. Suppose that $x \in X$ is such that $\eta_f(x) \in (\gamma\circ\lambda)_f^{-1}(U)$ and $\pi_U(\eta_f(x)) = b$, then $x \in \eta_f^{-1}((\gamma\circ\lambda)_f^{-1}(U)) = f^{-1}(U)$ by the previous lemma. Further we have $b = \pi_U(\eta_f(x)) = \pi_U(([x]_V)_{f(x)\in V\in\mathcal{O}(M)} = [x]_U$, hence $x \in b$. The converse follows from a similar argument.

Definition. Let $f\colon X \to M$ be continuous, then f is a *cosheaf space over* M if η_f is a homeomorphism.

By (Funk 1995, Theorem 5.9 and Remark 5.10) λ and γ form a pair of adjoint functors $\lambda \dashv \gamma$ with unit η. Further the counit ε for this adjunction is a natural isomorphism by (Funk 1995, Theorem 6.1). We summarize this as a

(1) **Theorem.** λ and γ form a pair of adjoint functors $\lambda \dashv \gamma$ with unit η and whose counit ε is an isomorphism.

Corollary. The category of cosheaves on M is equivalent to the reflective subcategory of cosheaf spaces over M.

Proof. This follows with (Gabriel and Zisman 1967, Proposition 1.3 or http://ncatlab.org/nlab/show/reflective+subcategory#characterizations).

Remark. Beyond the above Funk (1995 Theorem 5.17) provides a topological characterization of cosheaf spaces which hasn't been mentioned here.

The Reeb Space

de Silva, Munch, and Patel (2015) observed that $\gamma \circ \lambda$ is closely related to another endofunctor on topological spaces over M, the Reeb space.

> **Definition.** Given a continuous map $f \colon X \to M$ and $x \in X$ let $\pi_f(x)$ be the connected component of x in $f^{-1}(f(x))$. In this way we obtain a function $\pi_f \colon X \to 2^X$ and we endow $\pi_f(X)$ with the quotient topology[1]. By the universal property of the quotient space there is a unique continuous function $\tilde{f} \colon \pi_f(X) \to M$ such that $\tilde{f} \circ \pi_f = f$ and we define $\rho_f = \tilde{f}$.

With this definition ρ forms an endofunctor on topological spaces over M and π a natural transformation from id to ρ. Given a continuous map $f \colon X \to M$ for a locally connected topological space X the universal property of the quotient space induces a unique map $\phi_f \colon \pi_f(X) \to \operatorname{dis} \lambda_f$ such that $\phi_f \circ \pi_f = \eta_f$ and thus in particular $\rho_f = (\gamma \circ \lambda)_f \circ \pi_f$, hence we have the following commutative diagram

$$\begin{tikzcd} & \mathrm{id} \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ \rho \arrow[rr, "\phi"'] & & \gamma \circ \lambda \end{tikzcd}$$

in the category of endofunctors on locally connected topological spaces over M.

> **Proposition.** The natural transformation $\lambda \circ \phi$ from $\lambda \circ \rho$ to $\lambda \circ (\gamma \circ \lambda)$ is an isomorphism.
>
> *Proof.* We apply λ to the previous diagram and obtain
>
> $$\begin{tikzcd} & \lambda \arrow[dl, "\lambda\circ\pi"'] \arrow[dr, "\lambda\circ\eta"] & \\ \lambda \circ \rho \arrow[rr, "\lambda\circ\phi"'] & & \lambda \circ \gamma \circ \lambda. \end{tikzcd}$$
>
> Since $\lambda \circ \pi$ is an isomorphism, it suffices to show that $\lambda \circ \eta$ is an isomorphism. Given $f \colon X \to M$ we apply the inverse bijection induced by the adjunction

[1] This is in line with the previously made assumption, since quotient spaces of locally connected spaces are again locally connected.

$(\lambda \dashv \gamma, \eta)$ to the diagram

$$\begin{tikzcd}
f \arrow[dr, "\eta_f"] \arrow[dd, "\eta_f"'] & \\
& (\gamma \circ \lambda)_f \\
(\gamma \circ \lambda)_f \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

and obtain

$$\begin{tikzcd}
\lambda_f \arrow[dr, "\mathrm{id}"] \arrow[dd, "(\lambda \circ \eta)_f"'] & \\
& \lambda_f \\
(\lambda \circ \gamma \circ \lambda)_f, \arrow[ur, "(\varepsilon \circ \lambda)_f"'] &
\end{tikzcd}$$

hence $(\lambda \circ \eta)_f$ is the inverse to $(\varepsilon \circ \lambda)_f$.

(2) **Corollary.** ϕ and $\eta \circ \rho$ are naturally ismorphic as functors from the category of topological spaces over $\mathbb{R}$ to the category of homomorphisms in the category of topological spaces over $\mathbb{R}$.

Example. Let $f \colon X \to \mathbb{R}$ be a proper Morse function, then the critical points of f are isolated and since f is proper, it's critical values are isolated as well. Hence for each $r \in \mathbb{R}$ there is an $\varepsilon_r > 0$ such that for all $0 < \delta \leq \varepsilon_r$ the inclusion of $f^{-1}(r)$ into $f^{-1}((r - \delta, r + \delta))$ is a homotopy equivalence and thus ϕ_f is a homeomorphism.

de Silva, Munch, and Patel (2015) provide a self-contained treatment of the above when λ and γ are restricted to full subcategories of topological spaces over $\mathbb{R}$ respectively cosheaves on $\mathbb{R}$. When ϕ is restricted to this subcategory of topological spaces over $\mathbb{R}$ referred to as constructible $\mathbb{R}$-spaces, then ϕ is a natural isomorphism. Further the authors provide a geometric description of the resulting subcategory of cosheaf spaces over $\mathbb{R}$. They refer to this category as **Reeb** or as the category of $\mathbb{R}$-graphs.

Ascending Cosheaves

In addition to the space $\mathbb{R}$ we consider the reals augmented with a coarser topology.

Definition. Let $\bar{\mathbb{R}}$ be the topological space $(\mathbb{R}, \{(-\infty, r)\}_{-\infty \leq r \leq \infty})$, then we have the continuous map $\mathrm{id} \colon \mathbb{R} \to \bar{\mathbb{R}}, x \mapsto x$.

We can pushforward cosheaves on $\mathbb{R}$ to cosheaves on $\bar{\mathbb{R}}$ via id.

Definition. Given a cosheaf F on $\mathbb{R}$ and $-\infty \leq r \leq \infty$ we define $\mathrm{id}_* F((-\infty, r)) = F((-\infty, r))$.

Similar to defining the pullback for sheaves, we take two steps to define the pullback of a cosheaf via id.

Definition. Given a cosheaf F on $\bar{\mathbb{R}}$ and an open subset $U \subseteq \mathbb{R}$ we define $\mathrm{id}^+ F(U) := F((-\infty, \sup U))$.

With this definition $\mathrm{id}^+ F$ is merely a precosheaf for all we know. Yet we have the following.

Lemma. Given a cosheaf F on $\bar{\mathbb{R}}$ the precosheaf $\mathrm{id}^+ F$ is a cosheaf on the poset of open intervals.

Proof. Let $(a,b) = \bigcup_{i \in I}(a_i, b_i)$. Without loss of generality we assume that I has a linear order such that $b_i \leq b_j$ for all $i \leq j$ and $a_i \leq a_j$ for all $i \leq j$ with $b_i = b_j$. Since $b = \sup_{i \in I} \sup(a_i, b_i) = \sup_{i \in I} b_i$ and since F is a cosheaf we have the coequalizer diagram

$$\coprod_{i<j} F((-\infty, b_i)) \overset{\sigma}{\underset{\sigma'}{\rightrightarrows}} \coprod_{i \in I} F((-\infty, b_i)) \longrightarrow F((-\infty, b))$$

where $\sigma_{i,j}$ maps $F((-\infty, b_i))$ identical to $F((-\infty, b_i))$ and $\sigma'_{i,j}$ maps $F((-\infty, b_i))$ to $F((-\infty, b_j))$ via the induced inclusion. Now suppose we have $i < j$ such that $b_i \leq a_j$, then since $[b_i, a_j]$ is compact we can find $i < i_1 < ... < i_k$ such that $[b_i, a_j] \subseteq \bigcup_{l=1}^{k}(a_{i_l}, b_{i_l})$ and such that this cover is minimal. If $i_k < j$ we set $\tau = \sigma_{i_k,j}$, $\tau' = \sigma'_{i_k,j}$ and if $j < i_k$ we set $\tau = \sigma_{j,i_k}$, $\tau' = \sigma'_{j,i_k}$. With this we have $\sigma'_{i,j} \circ \sigma^{-1}_{i,j} = \tau' \circ \tau^{-1} \circ \sigma'_{i_{k-1},i_k} \circ \sigma^{-1}_{i_{k-1},i_k} \circ ... \circ \sigma'_{i_1,i_2} \circ \sigma^{-1}_{i_1,i_2} \circ \sigma'_{i,i_1} \circ \sigma^{-1}_{i,i_1}$ and yet at the same time $b_j > a_{i_k}$, $b_{i_k} > a_j$, $b_{i_{k-1}} > a_{i_k}$, $\dots$, $b_{i_1} > a_{i_2}$, and $b_i > a_{i_1}$, since $[b_i, a_j]$ is connected and the cover is minimal. Thus we may omit all terms from the leftmost coproduct in the above diagram where $b_i \leq a_j$ without loosing the property of it being a coequalizer diagram. Now for any $i < j$ such that $b_i > a_j$ we have $\mathrm{id}^+ F((a_i, b_i) \cap (a_j, b_j)) = F((-\infty, b_i))$, hence we may replace the corresponding term in the leftmost coproduct by $\mathrm{id}^+ F((a_i, b_i) \cap (a_j, b_j))$. Similarly we may replace the terms in the middle and the term on the right to arrive at a coequalizer diagram of the form

$$\coprod_{i<j,\, b_i > a_j} \mathrm{id}^+ F((a_i, b_i) \cap (a_j, b_j)) \rightrightarrows \coprod_{i \in I} \mathrm{id}^+ F((a_i, b_i)) \longrightarrow \mathrm{id}^+ F((a,b)).$$

Now for $i < j$ such that $b_i \leq a_j$ we have $\mathrm{id}^+ F((a_i, b_i) \cap (a_j, b_j)) = \mathrm{id}^+ F(\emptyset) = F(\emptyset) = \emptyset$ which does not contribute to the coequalizer and this implies the claim.

Definition. Given a cosheaf F on $\bar{\mathbb{R}}$ and an open subset $U \subseteq \mathbb{R}$ we define $\mathrm{id}^{-1} F(U) := \varinjlim_{(a,b) \subseteq U} \mathrm{id}^+ F((a,b))$.

By the previous lemma $\mathrm{id}^{-1} F$ as defined above is a cosheaf.

Remark. $\mathrm{id}^{-1} F$ as defined above is isomorphic to the cosheafification of $\mathrm{id}^{+} F$ or the cosheaf associated to $\mathrm{id}^{+} F$, see for example (Funk 1995, Theorem 6.3 and Remark 6.4).

Definition. Given a cosheaf F on $\mathbb{R}$ we define a homomorpism η'_F of cosheaves from F to $\mathrm{id}^{-1} \mathrm{id}_* F$. Since both are cosheaves it suffices to define η'_F on open intervals. So for $-\infty \leq a < b \leq \infty$ we define η'_F from $F((a,b))$ to $\mathrm{id}^{-1} \mathrm{id}_* F((a,b)) = \mathrm{id}^{+} \mathrm{id}_* F((a,b)) = F((-\infty,b))$ to be the map induced by the inclusion $(a,b) \subseteq (-\infty,b)$.

Definition. Let F be a cosheaf on $\mathbb{R}$, then F is *ascending* if η'_F is an isomorphism.

(3) **Proposition.** id_* and id^{-1} form a pair of adjoint functors $\mathrm{id}_* \dashv \mathrm{id}^{-1}$ with unit η' and whose counit ε' is an isomorphism.

Proof. Let F be a cosheaf on $\mathbb{R}$, let G be a cosheaf on $\bar{\mathbb{R}}$, and let g be a homomorphism from F to $\mathrm{id}^{-1} G$. Now suppose we have a morphism f from $\mathrm{id}_* F$ to G such that $(\mathrm{id}^{-1} f) \circ \eta'_F = g$, then for any $r \in \mathbb{R}$ we have $g_{(-\infty,r)} = \big((\mathrm{id}^{-1} f) \circ \eta'_G\big)_{(-\infty,r)} = f_{(-\infty,r)}$ and this determines f. Now suppose f is defined by $g_{(-\infty,r)} = f_{(-\infty,r)}$ for any $r \in \mathbb{R}$ and we have $-\infty \leq a < b \leq \infty$, then $g_{(a,b)}$ is the same as $g_{(-\infty,b)}$ pre-composed with the map induced by inclusion from $F((a,b))$ to $F((-\infty,b))$ by naturality. But this is the same as $\big((\mathrm{id}^{-1} f) \circ \eta'_G\big)_{(a,b)}$ by definition of f, hence g and $(\mathrm{id}^{-1} f) \circ \eta'_G$ agree on a basis of $\mathbb{R}$.

By the above argument ε'_G is equal to $\mathrm{id}_{\mathrm{id}^{-1} G}$ when restricted to $\mathrm{id}_* \mathrm{id}^{-1} G((-\infty,r)) = \mathrm{id}^{-1} G((-\infty,r)) = G((-\infty,r))$, hence ε'_G is an isomorphism.

Corollary. The category of cosheaves on $\bar{\mathbb{R}}$ is equivalent to the reflective subcategory of ascending cosheaves on $\mathbb{R}$.

Proof. This follows with (Gabriel and Zisman 1967, Proposition 1.3 or http://ncatlab.org/nlab/show/reflective+subcategory#characterizations).

Ascending Spaces

Later we will make the ascending cosheaf $\mathrm{id}^{-1} \mathrm{id}_* \lambda_f$ for a continuous function f the cosheaf version of the join tree associated to f. As an intermediate step we show that we can obtain this cosheaf not only by post-composing λ with $\mathrm{id}^{-1} \mathrm{id}_*$ but also by pre-composing λ with another functor, the epigraph. This use of the epigraph in defining the join tree[2] is due to Morozov, Beketayev, and Weber (2013).

Definition. Let $f \colon X \to \mathbb{R}$ be a continuous map, it's *epigraph* is $\operatorname{epi} f := \{(x,y) \in X \times \mathbb{R} \mid y \geq f(x)\}$. Further we define $\iota_f \colon \operatorname{epi} f \to \mathbb{R}, (x,y) \mapsto y$ and $\kappa_f \colon X \to \operatorname{epi} f, x \mapsto (x, f(x))$.

[2]Though join trees are referred to as merge trees in the cited paper.

With these definitions ι defines a functor on topological spaces over $\mathbb{R}$ with κ a natural transformation from id to ι.

(4) **Definition.** A function $f\colon X \to \mathbb{R}$ is *ascending* if for all $r \in \mathbb{R}$ there is a continuous map $H_r\colon X \times [0,1] \to X$ such that $H_r(x,t) = x$ for all $0 \le t \le 1$ and $x \in X$ with $f(x) \ge r$ and such that $f(H_r(x,t)) = r + t(f(x) - r)$ for all $0 \le t \le 1$ and $x \in X$ with $f(x) \le r$.

(5) **Lemma.** For any continuous function $f\colon X \to \mathbb{R}$ the projection $\iota_f\colon \operatorname{epi} f \to \mathbb{R}$ is ascending.

Proof. For $r \in \mathbb{R}$ we set $H_r\colon \operatorname{epi} f \times [0,1] \to \operatorname{epi} f, ((x,y),t) \mapsto (x, \max\{r + t(y - r), y\})$.

(6) **Lemma.** For any ascending function $f\colon X \to \mathbb{R}$ the cosheaf λ_f is ascending as well.

Proof. Given $-\infty \le a < r < b \le \infty$ we proof that the maps from $\Lambda(f^{-1}([r,b)))$ to $\Lambda(f^{-1}((a,b))) = \lambda_f((a,b))$ respectively $\Lambda(f^{-1}((-\infty,b))) = \lambda_f((-\infty,b))$ induced by the inclusions are bijections[3]. From this our claim follows. Since inclusions as maps of spaces always commute the two bijections commute with $(\eta' \circ \lambda)_f$ as well, hence $(\eta' \circ \lambda)_f$ is a bijection as a map from $\lambda_f((a,b))$ to $\lambda_f((-\infty,b)) = \mathrm{id}_* \lambda_f((-\infty,b)) = \mathrm{id}^+ \mathrm{id}_* \lambda_f((a,b)) = \mathrm{id}^{-1} \mathrm{id}_* \lambda_f((a,b))$. And since the open intervals of $\mathbb{R}$ form a basis, the lemma follows.

Given any point $x \in f^{-1}((a,r))$ the map $t \mapsto H_r(x,t)$ defines a continuous path in $f^{-1}((a,b))$ from $H_r(x,0) \in f^{-1}([r,b))$ to x, hence induced map from $\Lambda(f^{-1}([r,b)))$ to $\Lambda(f^{-1}((a,b))) = \lambda_f((a,b))$ is surjective. Now suppose $x, y \in f^{-1}([r,b))$ lie in the same connected component C of $f^{-1}((a,b))$, then $H_r(C,0)$ is connected since H_r is continuous. Further $x, y \in H_r(C,0)$, hence the induced map from $\Lambda(f^{-1}([r,b)))$ to $\lambda_f((a,b))$ is injective. The induced map from $\Lambda(f^{-1}([r,b)))$ to $\lambda_f((-\infty,b))$ is a bijection by a similar argument.

Remark. The previous result remains valid if instead of λ we consider the pushforward of another cosheaf on X that maps inclusions of open sets in X that are homotopy equivalences to bijections of sets.

(7) **Lemma.** Given a continuous map $f\colon X \to \mathbb{R}$ the homomorphism $\mathrm{id}_*(\lambda \circ \kappa)_f$ from $\mathrm{id}_* \lambda_f$ to $\mathrm{id}_*(\lambda \circ \iota)_f$ is an isomorphism.

Proof. Given $b \in \mathbb{R} \cup \{\infty\}$ we show that $\kappa_f(f^{-1}((-\infty,b)) = \{(x, f(x))\}_{\{x \in X \mid f(x) < b\}}$ is a strong deformation retract of $\iota_f^{-1}((-\infty,b)) = \{(x,y) \in X \times (-\infty,b) \mid y \ge f(x)\}$. Then the result follows by a similar argument as the previous lemma. We define $R\colon \iota_f^{-1}((-\infty,b)) \times [0,1] \to \iota_f^{-1}((-\infty,b)), ((x,y),t) \mapsto (x, f(x) + t(y - f(x)))$, then $R((x,y),1) = (x,y)$ and $R((x,y),0) = (x, f(x))$ for all $(x,y) \in \iota_f^{-1}((-\infty,b))$.

[3] The space $f^{-1}([r,b))$ may not be locally connected. However we won't need this property.

(8) **Proposition.** The natural transformations $\eta' \circ \lambda$ and $\lambda \circ \kappa$ are isomorphic as objects in the category of functors from topological spaces over $\mathbb{R}$ to cosheaves on $\mathbb{R}$ under λ. In particular $\mathrm{id}^{-1}\,\mathrm{id}_*\,\lambda$ and $\lambda \circ \iota$ are naturally isomorphic.

Proof. Given $f\colon X \to \mathbb{R}$ we have the commutative diagram

$$\begin{tikzcd}
\lambda_f \arrow[r, "(\lambda\circ\kappa)_f"] \arrow[d, "(\eta'\circ\lambda)_f"'] & (\lambda\circ\iota)_f \arrow[d, "(\eta'\circ\lambda\circ\iota)_f"] \\
\mathrm{id}^{-1}\,\mathrm{id}_*\,\lambda_f \arrow[r, "\mathrm{id}^{-1}\,\mathrm{id}_*(\lambda\circ\kappa)_f"'] & \mathrm{id}^{-1}\,\mathrm{id}_*(\lambda\circ\iota)_f.
\end{tikzcd}$$

By lemma 5 and lemma 6 the homomorphism $(\eta' \circ \lambda \circ \iota)_f$ is an isomorphism. And by lemma 7 we have that $\mathrm{id}^{-1}\,\mathrm{id}_*(\lambda \circ \kappa)_f$ is an isomorphism.

The Join Tree

The following definition[4] is from (Morozov, Beketayev, and Weber 2013).

Definition. Let $f\colon X \to \mathbb{R}$ be a continuous map we define it's *join tree* to be the continuous map $(\rho \circ \iota)_f$ from $(\pi \circ \iota)_f(\operatorname{epi} f)$ to $\mathbb{R}$.

With this definition $\rho \circ \iota$ is an endofunctor on topological spaces over $\mathbb{R}$. Given a continuous map $f\colon X \to \mathbb{R}$ we have $(\pi \circ \iota)_f \circ \kappa_f = (\rho \circ \kappa)_f \circ \pi_f$, so in somewhat sloppy notation $(\pi \circ \iota) \circ \kappa = (\rho \circ \kappa) \circ \pi$ is a natural transformation from id to $\rho \circ \iota$. Similarly we have the function $(\gamma \circ \lambda \circ \iota)_f$ defined on the display space $\mathrm{dis}(\lambda \circ \iota)_f$ of $(\lambda \circ \iota)_f$. And just as with ρ we have the natural transformation $(\eta \circ \iota) \circ \kappa = (\gamma \circ \lambda \circ \kappa) \circ \eta$ from id to $\gamma \circ \lambda \circ \iota$. The two constructions are related via the commutative diagram

$$\begin{tikzcd}
 & \rho_f \arrow[r, "(\rho\circ\kappa)_f"] \arrow[d, "\phi_f"] & (\rho\circ\iota)_f \arrow[d, "(\phi\circ\iota)_f"] \\
f \arrow[ur, "\pi_f"] \arrow[dr, "\eta_f"'] & & \\
 & (\gamma\circ\lambda)_f \arrow[r, "(\gamma\circ\lambda\circ\kappa)_f"'] & (\gamma\circ\lambda\circ\iota)_f
\end{tikzcd}$$

given a function $f\colon X \to \mathbb{R}$. In the section on the Reeb space we considered the left triangle which suggests to replace the Reeb graph functor ρ and the natural transformation π by $\gamma \circ \lambda$ and η respectively. Now $\rho \circ \kappa$ yields a nice and classic map from any Reeb graph to the corresponding join tree, so our replacement of the Reeb graph functor ρ by $\gamma \circ \lambda$ is only complete, if also we can replace the join tree functor $\rho \circ \iota$ and the natural transformation $\rho \circ \kappa$ and if we can extend ϕ to a natural transformation from $\rho \circ \kappa$ to it's replacement. And here the commutative square on the right hand side, suggests we may take $\lambda \circ \gamma \circ \iota$ as a replacement for the join tree functor $\rho \circ \iota$ and to take $\gamma \circ \lambda \circ \kappa$ as a replacement for $\rho \circ \kappa$, since then we can extend ϕ by $\phi \circ \iota$ to a natural transformation from $\rho \circ \kappa$ to $\gamma \circ \lambda \circ \kappa$. We further note that by

[4]Though join trees are referred to as merge trees in the cited paper.

corollary 2 in that section the natural transformation $(\phi, \phi \circ \iota)$ from $\rho \circ \kappa$ to $\gamma \circ \lambda \circ \kappa$ is isomorphic to the natural transformation $(\eta \circ \rho, \eta \circ \rho \circ \iota)$ from $\rho \circ \kappa$ to $\gamma \circ \lambda \circ \rho \circ \kappa$, so our choice of replacements is the same as if we applied $\gamma \circ \lambda$ to the upper row in the diagram. And by proposition 8 we have a natural isomorphism from $(\gamma \circ \lambda \circ \iota)$ to $\gamma \operatorname{id}^{-1} \operatorname{id}_* \lambda$ that commutes with $\gamma \circ \lambda \circ \kappa$ and $\gamma \circ \eta' \circ \lambda$ sothat we can use the following

Proposition. $\operatorname{id}_* \lambda$ and $\gamma \operatorname{id}^{-1}$ form a pair of adjoint functors $\operatorname{id}_* \lambda \dashv \gamma \operatorname{id}^{-1}$ with unit $(\gamma \circ \eta' \circ \lambda) \circ \eta$ and whose counit is an isomorphism.

Proof. The first statement follows from theorem 1, proposition 3 and the general statement that the two pairs of adjoint functors, when composed in the same way as in our claim, form again a pair of adjoint functors with the unit described as in the claim, see for example https://en.wikipedia.org/wiki/Adjoint_functors#Composition. And for the counit of this composed adjunction we have the formula $\operatorname{id}_* \varepsilon \operatorname{id}^{-1} \circ \varepsilon'$. By theorem 1 ε is an isomorphism, hence $\operatorname{id}_* \varepsilon \operatorname{id}^{-1}$ is an isomorphism and by proposition 3 ε' is an isomorphism and thus our claim follows.

Corollary. The category of cosheaves on $\bar{\mathbb{R}}$ is equivalent to the reflective subcategory of ascending cosheaf spaces over $\mathbb{R}$.

Proof. By Gabriel and Zisman (1967 Proposition 1.3 or http://ncatlab.org/nlab/show/reflective+subcategory#characterizations) the category of cosheaves on $\bar{\mathbb{R}}$ is equivalent to the reflective subcategory of those spaces $f \colon X \to \mathbb{R}$ over $\mathbb{R}$ for which $(\gamma \circ \eta' \circ \lambda)_f \circ \eta_f$ is an isomorphism. Now suppose this is the case for f, then f is isomorphic to $\gamma \operatorname{id}^{-1} \operatorname{id}_* \lambda$ which is in the image of γ and thus a cosheaf space, hence η_f is an isomorphism. From this it follows that $(\gamma \circ \eta' \circ \lambda)_f$ is an isomorphism as well, hence by proposition 8 $(\gamma \circ \lambda \circ \kappa)_f$ is an isomorphism. Now we consider the commutative diagram

$$\begin{tikzcd}
f \arrow[r, "\kappa_f"] \arrow[d, "\eta_f"'] & \iota_f \arrow[d, "(\eta \circ \iota)_f"] \\
(\gamma \circ \lambda)_f \arrow[r, "(\gamma \circ \lambda \circ \kappa)_f"'] & (\gamma \circ \lambda \circ \iota)_f .
\end{tikzcd}$$

Hence we have the retract[5] $R := \eta_f^{-1} \circ (\gamma \circ \lambda \circ \kappa)_f^{-1} \circ (\eta \circ \iota)_f$ from ι_f to f. By lemma 5 ι_f is ascending, so given $r \in \mathbb{R}$ there is a map H_r as in definition 4. Now let $\tilde{H}_r \colon X \times [0,1] \to X$ be defined by $\tilde{H}_r(x,t) = r(H_r(\kappa_f(x), t))$ then $\tilde{H}_r$ inherits the properties needed in order for f to be ascending. Conversely if f is an ascending cosheaf space over $\mathbb{R}$, then η_f is an isomorphism since f is a cosheaf space. And by lemma 6 λ_f is ascending, hence $(\eta' \circ \lambda)_f$ is an isomorphism.

In conclusion $(\gamma \circ \lambda \circ \iota)_f$ is an ascending cosheaf space over $\mathbb{R}$ given a function f. It's cosheaf of connected components $(\lambda \circ \gamma \circ \lambda \circ \iota)_f$ is isomorphic to $(\lambda \circ \iota)_f$ by theorem 1. By lemma 5 and lemma 6 $(\lambda \circ \iota)_f$ is ascending, and thus we have an associated cosheaf $\operatorname{id}_*(\lambda \circ \iota)_f$ on $\bar{\mathbb{R}}$ via the adjunction $\operatorname{id}_* \dashv \operatorname{id}^{-1}$ by proposition 3. By lemma 7 this

[5] By a retract we mean a homomorphism R in the category of topological spaces over $\mathbb{R}$ from ι_f to f such that $R \circ \kappa_f = \operatorname{id}$.

cosheaf is isomorphic to $\mathrm{id}_* \lambda_f$ which is the cosheaf on $\bar{\mathbb{R}}$ associated to $\gamma\,\mathrm{id}^{-1}\,\mathrm{id}_*\,\lambda_f$ via the adjunction $\mathrm{id}_* \lambda \dashv \gamma\,\mathrm{id}^{-1}$. Now applying id^{-1} to $\mathrm{id}_* \lambda_f \cong \mathrm{id}_*(\lambda \circ \iota)_f$ recovers $(\lambda \circ \iota)_f$, hence $(\gamma \circ \lambda \circ \iota)_f$ and $\gamma\,\mathrm{id}^{-1}\,\mathrm{id}_*\,\lambda_f$ are isomorphic and thus a posteriori $\mathrm{id}_* \lambda_f$ is the cosheaf on $\bar{\mathbb{R}}$ associated to the ascending cosheaf space $(\gamma \circ \lambda \circ \iota)_f$ via the adjunction $\mathrm{id}_* \lambda \dashv \gamma\,\mathrm{id}^{-1}$. (Here the author allowed himself some redundance repeating the proof of proposition 8.)

From Sets to Algebras

For an integral domain A we consider the contravariant functor $\hom(\_, A)$ from the category of sets to the category of commutative unital A-algebras. We note that since A is an integral domain the idempotents of $\hom(L, A)$ for any set L are precisely the maps from L to A with values in $\{0, 1\}$.

(9) **Lemma.** $\hom(\_, A)$ is pseudomonic.

Proof. $\hom(\_, A)$ is faithful since for any map $m\colon L \to K$ and $k \in K$ we have $m^{-1}(k) = \big(\hom(m, A)(1_k)\big)^{-1}(1)$ where $1_k := 1_{\{k\}}$ and $1_{K'}$ is the indicator function for any subset $K' \subseteq K$.

Now suppose φ is an isomorphism from $\hom(K, A)$ to $\hom(L, A)$ then φ induces a bijection between the non-zero centrally primitive idempotents of $\hom(K, A)$ and $\hom(L, A)$. Now the non-zero centrally primitive idempotents of $\hom(K, A)$ are just the maps of the form 1_k for some $k \in K$ and similarly for $\hom(L, A)$. Let $m\colon L \to K$ be the corresponding inverse bijection, then for any $c \in \hom(K, A)$ and $l \in L$ we have

$$\begin{aligned}\varphi(c) \cdot 1_l &= \varphi(c \cdot 1_{m(l)}) = \varphi(c(m(l))1_{m(l)}) \\ &= c(m(l))\varphi(1_{m(l)}) = c(m(l))1_l \\ &= \hom(m, A)(c) \cdot 1_l\end{aligned}$$

and thus $\varphi = \hom(m, A)$.

Corollary. The functor $\hom(\_, A)$ induces an anti-equivalence between the category of sets and the replete image of $\hom(\_, A)$.

Corollary. For any category $\mathcal{C}$ the functor $\hom(\_, A)$ induces an anti-equivalence between the category of set-valued precosheaves on $\mathcal{C}$ and the category of presheaves with values in the replete image of $\hom(\_, A)$.

Lemma. $\hom(\_, A)$ is full when restricted to the category of finite sets.

Proof. Let $\varphi\colon \hom(K, A) \to \hom(L, A)$ be a homomorphism with K and L finite, then $\varphi(1_k)$ is an idempotent for each $k \in K$ and thus we have subsets $L_k \subseteq L$ such that $\varphi(1_k) = 1_{L_k}$. Further we have $\sum_{l \in L} 1_l = 1 = \varphi(1) = \varphi\big(\sum_{k \in K} 1_k\big) = \sum_{k \in K} \varphi(1_k) = \sum_{k \in K} 1_{L_k}$ and thus $L = \bigcup_{k \in K} L_k$. Now for any $k, k' \in K$ with $k \neq k'$ we have $0 = \varphi(0) = \varphi(1_k \cdot 1_{k'}) = 1_{L_k} \cdot 1_{L'_k}$, hence L_k and $L_{k'}$ are disjoint. Altogether we obtain that the subsets L_k with $k \in K$ form a partition of L and we may define a map $m\colon L \to K$ such that $m(l) = k$ for $l \in L_k$ for all $k \in K$. With this definition we have $\varphi = \hom(m, A)$ since the two maps agree on a basis of $\hom(K, A)$.

The following example shows that we cannot assume the unrestricted functor $\hom(\_, A)$ to be full, if A is a general ring.

Example. We consider $\hom(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})$. Let $\mathfrak{a}$ be the ideal of all $c \in \hom(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})$ with $c^{-1}(0)$ cofinite. By Krull's theorem $\hom(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})$ has a maximal ideal $\mathfrak{m}$ with $\mathfrak{a} \subset \mathfrak{m}$ and this gives a homomorphism of fields $i\colon \mathbb{Z}/p\mathbb{Z} \to \hom(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})/\mathfrak{m}$. We further have $[c]^p - [c] = [c^p - c] = 0$ for all $[c] \in \hom(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})/\mathfrak{m}$ and as $X^p - X$ is a polynomial of degree p it has at most p roots in $\hom(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})/\mathfrak{m}$ and thus i is a bijection. Now the canonical homomorphism from $\hom(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})$ to the quotient $\hom(\mathbb{N}, \mathbb{Z}/p\mathbb{Z})/\mathfrak{m}$ yields a homomorphism $\varphi\colon \hom(\mathbb{N}, \mathbb{Z}/p\mathbb{Z}) \to \hom(\{1\}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ which is not in the image of $\hom(\_, \mathbb{Z}/p\mathbb{Z})$, since for any map $m\colon \{1\} \to \mathbb{N}$ the element $1_{m(1)} \in \mathfrak{a} \subset \mathfrak{m}$ is mapped to $1 \in \mathbb{Z}/p\mathbb{Z}$ under $\hom(m, \mathbb{Z}/p\mathbb{Z})$.

Remark. From a discussion similar to that of the previous lemma and example we can conclude that for sets K and L with L non-empty, the map from $\hom(L, K)$ to $\hom_{A\text{-algebras}}(\hom(K, A), \hom(L, A))$ induced by $\hom(\_, A)$ is surjective if and only if all ideals[6] $\mathfrak{p}$ of $\hom(K, A)$, with $\hom(K, A)/\mathfrak{p} \cong A$ as A-algebras, are of the form $\{c \in \hom(K, A) \mid c(k) = 0\}$ for some $k \in K$.

Lemma. $\hom(\_, A)$ is continuous as a functor from the opposed category of sets to the category of A-algebras.

Proof. We argue that $\hom(\_, A)$ is continuous as a functor to the category of commutative rings, the lemma then follows by a general result about limits in the under category. We fix a small category D. For an object X of any category $\mathcal{C}$ we denote by $\Delta(X)$ the constant functor from D to $\mathcal{C}$ that maps any object of D to X and any morphism of D to the identity. Let F be a functor from D to the category of sets, then we have the canonical natural transformation $t\colon F \to \Delta(\operatorname{colim}(F))$. Now $\hom(\Delta(\operatorname{colim}(F)), A) = \Delta(\hom(\operatorname{colim}(F), A))$ and by the universal property of the limit of $\hom(F(\_), A)$ we have a homomorphism of rings $s\colon \lim(\hom(F(\_), A)) \to \hom(\operatorname{colim}(F), A)$ such that $(\hom(\_, A) \circ t) \circ \Delta(s)$ is the canonical natural transformation from $\Delta(\lim(\hom(F(\_), A)))$ to $\hom(F(\_), A)$. Now the forgetful functor from the category of commutative rings to the category of sets is continuous as well as $\hom(\_, A)$ as a functor to the category of sets, hence in the category of sets both $(\hom(\_, A) \circ t) \circ \Delta(s)$ and $\hom(\_, A) \circ t$ itself satisfy the universal property of the limit of $\hom(F(\_), A)$, and thus s is a bijection.

Corollary. If D is a set-valued cosheaf, then $\hom(D(\_), A)$ defines a sheaf with values in the category of A-algebras.

Example. For any locally path connected topological space X the singular homology $H_0(X)$ is naturally isomorphic to the free abelian group with basis $\Lambda(X)$ and by the universal property of the free ablian group the restriction from $\hom_{\mathbb{Z}}(H_0(X), A)$ to $\hom(\Lambda(X), A)$ is an isomorphism of A-modules. Further we have a natural isomorphism of A-modules from $H^0(X, A)$ to $\hom_{\mathbb{Z}}(H_0(X), A)$ by the universal coefficient theorem and since for any $x \in X$ and $\alpha, \beta \in H^0(X, A)$

[6] which are prime necessarily

we have

$$\begin{aligned}\langle \alpha \cup \beta, [x] \rangle &= \langle H^0(d, A)(\alpha \times \beta), [x] \rangle = \langle \alpha \times \beta, H_0(d)([x]) \rangle \\ &= \langle \alpha \times \beta, [(x, x)] \rangle = \langle \alpha \times \beta, [x] \times [x] \rangle \\ &= \langle \alpha, [x] \rangle \langle \beta, [x] \rangle,\end{aligned}$$

where $d\colon X \to X \times X, x \mapsto (x, x)$ is the diagonal map, the composition of these two isomorphisms is an ismorphism of A-algebras. Since the above identifications are natural in X, the functors $\hom(\Lambda(\_), A)$ and $H^0(\_, A)$ define isomorphic sheaves on any locally path connected topological space.

Given a continuous function $f\colon X \to M$ from a locally path connected topological space X to M, the sheaves $f_* \hom(\Lambda(\_), A) \cong f_* H^0(\_, A)$ and $\hom(\lambda_f(\_), A)$ are identical. Bubenik, de Silva, and Scott (2014) define a generalized persistence module on the poset of open sets of M to be a functor to another category, thus λ_f is a generalized persistence module with values in the opposed category of sets and $f_* H^0(\_, A)$ is a persistence module with values in the category of A-algebras. A functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ then gives rise to a map from the generalized persistence modules with values in $\mathcal{C}$ to persistence modules with values in $\mathcal{D}$, so in their language $f_* H^0(\_, A)$ is the image of λ_f under the map induced by $\hom(\_, A)$ and thus their theory can be used to relate these two constructions in the context of topological persistence.

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