Generic 1D-Interleavings

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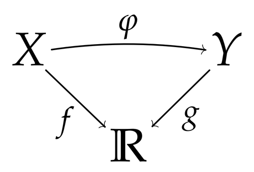
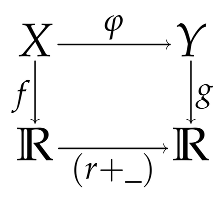
Introduction

Suppose $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ are continuous functions. Then we may ask the following two questions:

- Is there a homeomorphism $\varphi\colon X \to Y$ such that



  commutes? (In the following we call φ and any other continuous map that fits into the above diagram a *homomorphism* from f to g. This augments the class of real-valued continuous functions with the structure of a category, the *category of* $\mathbb{R}$*-spaces* for short.)
- Is there a homeomorphism $\varphi\colon X \to Y$ and a real number r such that



  commutes? (In other words $r + f(p) = g(\varphi(p))$ for all $p \in X$.)

If the answer to either of the above questions is no, we may still ask a weaker question. Instead of commutativity of the above diagrams, we only ask for commutivity up to ε for some $\varepsilon \geq 0$. For $\varepsilon \geq 0$ we have the two questions:

- Is there a homeomorphism $\varphi\colon X \to Y$ such that for all $p \in X$ the estimates $-\varepsilon \leq f(p) - g(\varphi(p)) \leq \varepsilon$ hold?
- Is there a homeomorphism $\varphi\colon X \to Y$ and a real number r such that for all $p \in X$ the estimates $-\varepsilon \leq r + f(p) - g(\varphi(p)) \leq \varepsilon$ hold?

Now both these questions depend on the parameter $\varepsilon \geq 0$ and for either question we may ask how large does ε need to be in order for the answer to be *yes*.

Definition. *Let $M(f,g)$ be the infimum of all $\varepsilon \geq 0$ such that the answer to the first question is affirmative. We name this* the absolute distance of f and g.
And let $\mu(f,g)$ be the infimum of all $\varepsilon \geq 0$ such that the answer to the second question is affirmative. We name this the relative distance of f and g.

In the very first question we ask whether f and g are isomorphic in the category of $\mathbb{R}$-spaces. Thus we may apply a functor F from the category of $\mathbb{R}$-spaces to a category in which it is easier to decide whether two objects are isomorphic or not. If $F(f)$ and $F(g)$ are isomorphic, then we still don't know. But if $F(f)$ and $F(g)$ are not isomorphic, then we know the answer is *no*. Similarly we may never learn the absolute or relative distance of f and g but we may try to find lower bounds to either value. To this end functors from the category of $\mathbb{R}$-spaces may be useful, but this concept alone doesn't solve our problem. In the following we describe a framework for *enhancing* functors from the category of $\mathbb{R}$-spaces to obtain lower bounds for $M(f,g)$ and $\mu(f,g)$. On several aspects this borrows from [BdS14].
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Monoidal Posets for 1D-Interleavings

Definition. *Let $D := \{(a,b) \in [-\infty,\infty) \times (-\infty,\infty] \mid a \leq b\}$ then D is a monoid by component-wise addition. Moreover we set $\mathbf{o} := (0,0)$ and we define a partial order $\preceq$ on D by*

$$(x,y) \preceq (x',y') \quad \text{if and only if} \quad x \geq x' \text{ and } y \leq y'.$$

A set augmented with a partial ordering and a monoid structure like D, we call a *monoidal poset*. Moreover $D \times D$ is a monoidal poset with the product ordering and component-wise addition. With some abuse of notation we write $(a,b;c,d)$ for points $((a,b),(c,d)) \in D \times D$. Now we define three monoidal sub-posets of $D \times D$.

Definition. *We set*

- $\mathcal{D} := \{(a,b;c,d) \in D \times D \mid a + c \leq 0 \leq b + d\}$,
- $\triangledown := \{(a,b;c,d) \in \mathcal{D} \mid c + b = 0 = a + d\}$, *and*
- $\blacktriangledown := \{(a,b;c,d) \in \triangledown \mid a + b = 0\}$.

Both the inclusion of $\triangledown$ into $\mathcal{D}$ and the inclusion of $\blacktriangledown$ into $\triangledown$ each have a lower adjoint in the sense of monotone Galois connections.

Definition. *Let $\delta\colon \mathcal{D} \to \triangledown$ be the lower adjoint of $\triangledown \subset \mathcal{D}$ and let $\gamma\colon \triangledown \to \blacktriangledown$ be the lower adjoint of $\blacktriangledown \subset \triangledown$.*

We note that both δ and γ are sub-linear, i.e. oplax monoidal as functors. We use these maps to describe two weightings on $\mathcal{D}$.

Definition. *We set $\epsilon\colon \triangledown \to [0,\infty]$, $(a,b;c,d) \mapsto \frac{1}{2}(b+d)$.*

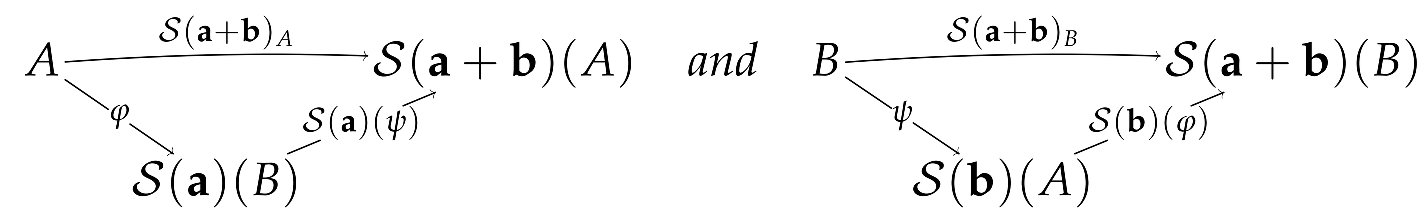
We note that ϵ is a (strict) homomorphism of monoidal posets. Moreover the restriction $\epsilon|_{\blacktriangledown}$ is an isomorphism. The compositions $\epsilon \circ \gamma \circ \delta$ and $\epsilon \circ \delta$ yield two different weightings on $\mathcal{D}$. It turns out $(\epsilon \circ \gamma \circ \delta)((a,b;c,d)) = \max\{-a,b,-c,d\}$ and $(\epsilon \circ \delta)((a,b;c,d)) = \frac{1}{2}(\max\{-c,b\} + \max\{-a,d\})$ for all $(a,b;c,d) \in \mathcal{D}$.

Interleavings in D-modules

Definition (D-modules)**.** *A D-module is a category $\mathcal{C}$ with a strict monoidal functor $\mathcal{S}$ from D to the category of endofunctors on $\mathcal{C}$. We refer to $\mathcal{S}$ as the* smoothing functor *of $\mathcal{C}$.*

Now let $\mathcal{C}$ be a D-module with smoothing functor $\mathcal{S}$. For $a \leq 0$, $b \geq 0$, and an object A in $\mathcal{C}$ we get two things, the image $\mathcal{S}((a,b))(A)$ of A under the endofunctor $\mathcal{S}((a,b))$ and a natural homomorphism from A to $\mathcal{S}((a,b))(A)$ induced by $\mathbf{o} \preceq (a,b)$. With some abuse of notation we denote this homomorphism by $\mathcal{S}((a,b))_A$. Now let A and B be objects in $\mathcal{C}$.

Definition (Interleavings)**.** *For $(\mathbf{a},\mathbf{b}) \in \mathcal{D}$ an* $(\mathbf{a},\mathbf{b})$-interleaving of A and B *is a pair of homomorphisms $\varphi\colon A \to \mathcal{S}(\mathbf{a})(B)$ and $\psi\colon B \to \mathcal{S}(\mathbf{b})(A)$ such that*



commute.
We say A and B are $(\mathbf{a},\mathbf{b})$-interleaved *if there is an $(\mathbf{a},\mathbf{b})$-interleaving of A and B.*

Now we use the weightings on $\mathcal{D}$ described on the previous card to describe two notions of an interleaving distance.

Definition. *Let $\mathcal{I}$ be the set of all $(\mathbf{a},\mathbf{b}) \in \mathcal{D}$ such that there is an $(\mathbf{a},\mathbf{b})$-interleaving of A and B. Then we set*

$$M_{\mathcal{S}}(A,B) := \inf(\epsilon \circ \gamma \circ \delta)(\mathcal{I}) \quad \text{and} \quad \mu_{\mathcal{S}}(A,B) := \inf(\epsilon \circ \delta)(\mathcal{I}).$$

We name $M(A,B)$ the absolute interleaving distance of A and B *and $\mu(A,B)$* the relative interleaving distance.

Now let $\mathcal{C}'$ be another D-module with smoothing functor $\mathcal{S}'$.

Definition. *A 1-homomorphism of D-modules from $\mathcal{C}$ to $\mathcal{C}'$ is a functor from $\mathcal{C}$ to $\mathcal{C}'$ such that*

$$G \circ \mathcal{S}(\mathbf{a}) = \mathcal{S}'(\mathbf{a}) \circ G \quad \text{and} \quad G \circ \mathcal{S}(\mathbf{a} \preceq \mathbf{b}) = \mathcal{S}'(\mathbf{a} \preceq \mathbf{b}) \circ G$$

for all $\mathbf{a},\mathbf{b} \in D$ with $\mathbf{a} \preceq \mathbf{b}$.

Theorem. *Let G be a 1-homomorphism from $\mathcal{C}$ to $\mathcal{C}'$, then*

$$M_{\mathcal{S}'}(G(A),G(B)) \leq M_{\mathcal{S}}(A,B) \quad \text{and} \quad \mu_{\mathcal{S}'}(G(A),G(B)) \leq \mu_{\mathcal{S}}(A,B).$$

Persistence-enhanced Functors

Definition. *Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions and let $(a,b),(c,d) \in D$. Then a* homomorphism from $(f,(a,b))$ to $(g,(c,d))$ *is a continuous map $\varphi\colon X \to Y$ such that $c - a \leq f(p) - g(\varphi(p)) \leq d - b$ for all $p \in X$. This defines a category which we denote by $\mathcal{F}$.*

Next we define a smoothing functor for $\mathcal{F}$.

Definition. *Let $f\colon X \to \mathbb{R}$ be a continuous function and let $\mathbf{a},\mathbf{b},\mathbf{c} \in D$ with $\mathbf{b} \preceq \mathbf{c}$. Then we set $\mathcal{T}(\mathbf{b})((f,\mathbf{a})) := (f,\mathbf{a}+\mathbf{b})$ and $\mathcal{T}(\mathbf{b} \preceq \mathbf{c})_{(f,\mathbf{a})} := \mathrm{id}_X$.*

Lemma. *For two $\mathbb{R}$-spaces f and g we have*

$$M(f,g) = M_{\mathcal{T}}((f,\mathbf{o}),(g,\mathbf{o})) \quad \text{and} \quad \mu(f,g) = \mu_{\mathcal{T}}((f,\mathbf{o}),(g,\mathbf{o})).$$

Now let F be a functor from the category of $\mathbb{R}$-spaces to a category $\mathcal{C}$.

Definition. *A* persistence-enhancement of F *is the structure of a D-module on $\mathcal{C}$ together with a 1-homomorphism $\tilde{F}$ from $\mathcal{F}$ to $\mathcal{C}$ such that $\tilde{F}((\_,\mathbf{o})) = F$.*

The previous lemma and the theorem from the previous card have the following

Corollary. *If there is a persistence-enhancement of F with smoothing functor $\mathcal{S}$, then we have*

$$M_{\mathcal{S}}(F(f),F(g)) \leq M(f,g) \quad \text{and} \quad \mu_{\mathcal{S}}(F(f),F(g)) \leq \mu(f,g)$$

for all $\mathbb{R}$-spaces f and g.

Finally we note that the *interleaving distances* provided in [Cha+09], [MBW13], [BS14], and [dMP16] can each be realized as the absolute interleaving distance of a corresponding persistence-enhanced functor.

References

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