The Mayer-Vietoris Pyramid Sheaf-theoretically

SFB TRR 109 Discretization in Geometry and Dynamics

In the following let k be a field and let $\mathbf{D}^b(k_{\mathbb{R}})$ be the bounded derived category of sheaves on $\mathbb{R}$ with values in the category of k-vector spaces.

1 A Happel Functor

We consider the γ-topology on $\mathbb{R}^2$, where $\gamma := [0,\infty) \times (-\infty, 0]$. Setting

$$q(a,b) := (a,\infty) \times (-\infty, b)$$

we have the following base

$$\mathcal{B} := \{q(a,b) \mid a,b \in \mathbb{R}\}.$$

Moreover we have the homeomorphism

$$T\colon \mathbb{R}^2_\gamma \to \mathbb{R}^2_\gamma, (x,y) \mapsto (-y-\pi, -x+\pi).$$

Now T yields an automorphism of $\mathcal{B}$ which we also denote by T. We aim to define a functor

$$\iota\colon \mathcal{B} \to \mathbf{D}^b(k_{\mathbb{R}})$$

such that

$$\iota \circ T = \iota(\_)[1]. \tag{1}$$

We start by setting

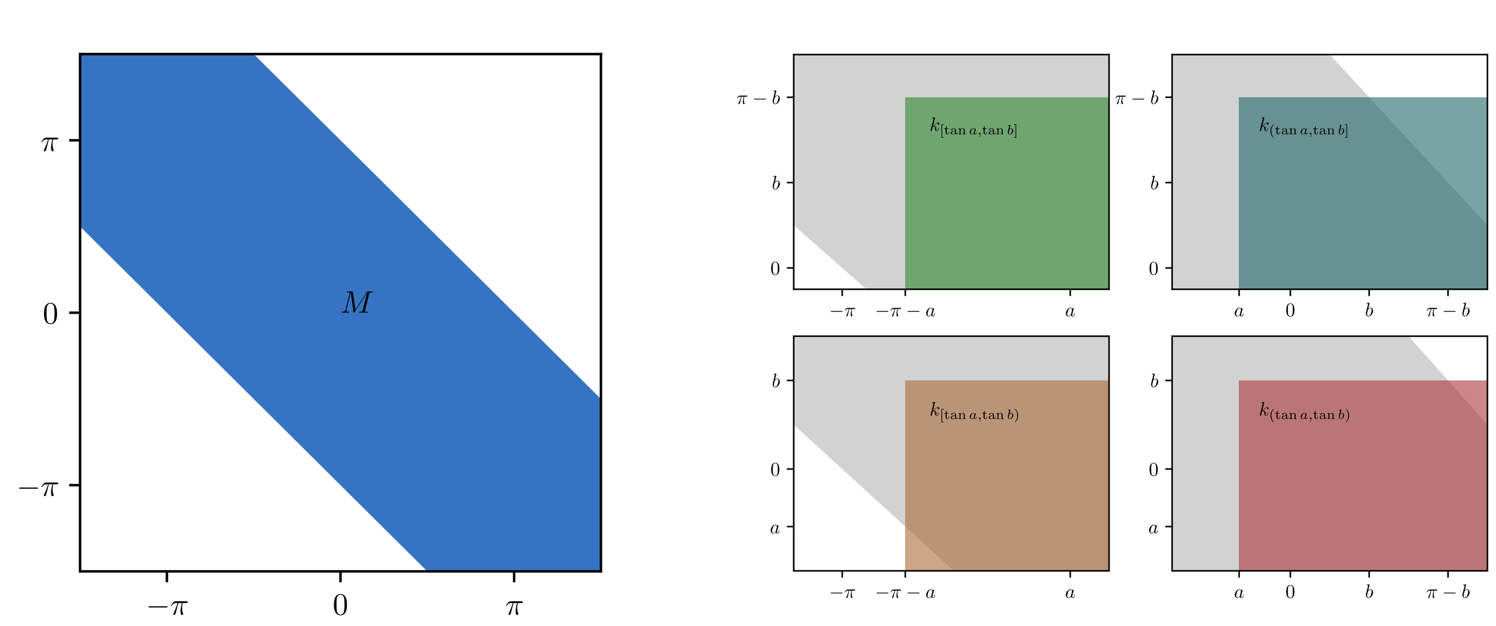
$$\iota(q(a,b)) := 0 \quad \text{for all } \ a,b \in \mathbb{R} \ \text{ with } \ |a+b| \geq \pi.$$

In some sense ι is supported on $M := \{(x,y) \in \mathbb{R}^2 \mid -\pi \leq x+y \leq \pi\}$.

For $-\frac{\pi}{2} \leq a \leq b \leq \frac{\pi}{2}$ we set

$$\begin{aligned} \iota(q(a,b)) &:= k_{(\tan a, \tan b)} \\ \iota(q(-\pi-a,b)) &:= k_{[\tan a, \tan b)} \\ \iota(q(a,\pi-b)) &:= k_{(\tan a, \tan b]} \\ \iota(q(-\pi-a,\pi-b)) &:= k_{[\tan a, \tan b]} \end{aligned}$$

whenever it makes sense.



So we specified ι on

$$\mathcal{U} := \{q(a,b) \mid a,b \in \mathbb{R}, 0 < b-a \leq 2\pi\}.$$

For $U,V \in \mathcal{U}$ with $U \subseteq V$ we choose $\iota(U \subseteq V)$ in the canonical way. Together with condition (1) this almost determines ι. It remains to specify for $\frac{\pi}{2} \leq a \leq b \leq c \leq \frac{\pi}{2}$ (whenever it makes sense) homomorphisms

$$\begin{aligned} k_{[\tan b, \tan c]} &= \iota(q(-\pi-b, \pi-c)) \to \iota(T(q(a,b))) = k_{(\tan a, \tan b)}[1] \\ k_{[\tan a, \tan b]} &= \iota(q(-\pi-a, \pi-b)) \to \iota(T(q(b,c))) = k_{(\tan b, \tan c)}[1]. \end{aligned}$$

We specify these maps in the form of extensions:

$$0 \to k_{(\tan a, \tan b)} \to k_{(\tan a, \tan c]} \to k_{[\tan b, \tan c]} \to 0$$
$$0 \to k_{(\tan b, \tan c)} \to k_{[\tan a, \tan c)} \xrightarrow{-1} k_{[\tan a, \tan b]} \to 0.$$

2 The Mayer-Vietoris Presheaf

Let $f\colon X \to \mathbb{R}$ be a continuous function, then

$$\hom(\iota\_, Rf_* k_X)$$

defines a presheaf supported on M. More generally we have the functor

$$h\colon \mathbf{D}^b(k_{\mathbb{R}}) \to \mathfrak{PSh}(\mathcal{B}), F \mapsto \hom(\iota\_, F)$$

to the category of presheaves supported on M.

2.1 Interleavings

For convenience we set

$$\alpha\colon \mathbb{R} \to S^1, x \mapsto \frac{1}{\sqrt{x^2+1}}(1,x)$$

and

$$p\colon \mathbb{R} \to S^1, t \mapsto e^{it}.$$

For $a,b \in \mathbb{R}$ let

$$\bar{s}^{(a,b)}\colon S^1 \to S^1, (x,y) \mapsto \begin{cases} (x,y), & x = 0 \\ \alpha\left(\frac{y}{x}+a\right), & x > 0 \\ -\alpha\left(\frac{y}{x}-b\right), & x < 0 \end{cases}$$

and let $s^{(a,b)}\colon \mathbb{R} \to \mathbb{R}$ be the unique continuous map with

$$s^{(a,b)}\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad \text{and} \quad \bar{s}^{(a,b)} \circ p = p \circ s^{(a,b)}.$$

With this we set

$$S^{(a,b)}\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto \left(s^{(a,b)}(x), s^{(b,a)}(y)\right)$$

and we note that

$$S^{(a,b)} \circ T = T \circ S^{(a,b)}.$$

Altogether we get a homomorphism of monoidal posets

$$\mathbb{R}^\circ \times \mathbb{R} \to \operatorname{Aut}(\mathcal{B}), (a,b) \mapsto S^{(a,b)}$$

and post-composing this with the contravariant strict monoidal functor

$$\operatorname{Aut}(\mathcal{B}) \to \operatorname{Aut}(\mathfrak{PSh}(\mathcal{B})), \begin{cases} T \mapsto (\_) \circ T \\ \eta \mapsto (\_) \circ \eta \end{cases}$$

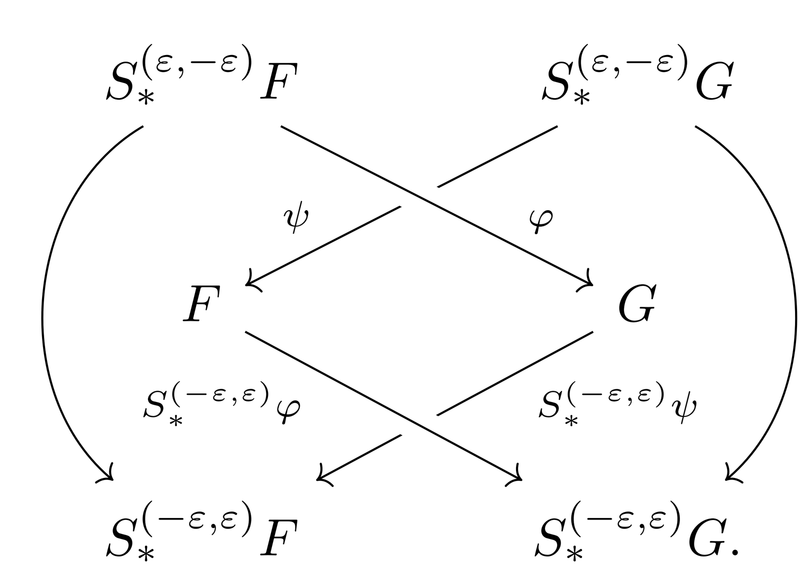
we get a contravariant strict monoidal functor

$$\mathcal{S}\colon \mathbb{R}^\circ \times \mathbb{R} \to \operatorname{Aut}(\mathfrak{PSh}(\mathcal{B}))$$

with

$$\mathcal{S}(a,b)(F) = S_*^{-(a,b)} F \quad \text{for all } \ F \in \mathfrak{PSh}(\mathcal{B}) \ \text{ and } \ a,b \in \mathbb{R}.$$

Analogously to (de Silva, Munch, and Stefanou 2017) and (Fluhr 2017) we can define ε-interleavings of the form



3 Constructibility

Definition *(Flip-Grid-constructible Presheaves on $\mathcal{B}$)*

> A presheaf F on $\mathcal{B}$ is *flip-grid-constructible* if it is point-wise finite-dimensional, has finite support, and if there is a finite subset $C \subset S^1$ such that $F(U \subseteq V)$ is an isomorphism for all $U,V \in \mathcal{B}$ with $U \cap \tilde{C} = V \cap \tilde{C}$, where $\tilde{C} := p^{-1}(C) \times \left(-p^{-1}(C)+\pi\right)$.

Now a flip-grid-constructible presheaf is a sheaf and the category of flip-grid-constructible sheaves supported on M is an Abelian Frobenius category which we denote by $\mathbf{C}$. By Kashiwara and Schapira (2017, Corollary 1.20 and Proposition 1.16) the subcategory $\mathbf{D}^b_{\mathbb{R}c,c}(k_{\mathbb{R}}) \subset \mathbf{D}^b(k_{\mathbb{R}})$ is in the full additive subcategory generated by the image of ι. Thus h restricts to a full and faithful functor from $\mathbf{D}^b_{\mathbb{R}c,c}(k_{\mathbb{R}})$ to the category of projectives in $\mathbf{C}$. In particular

$$h(Rf_* k_X) = \hom(\iota\_, Rf_* k_X)$$

is projective for a Morse function $f\colon X \to \mathbb{R}$ defined on a closed smooth manifold X.