Contents

Introduction

The topic of the present text is the interleaving distance of join trees by Morozov, Beketayev, and Weber (2013). Just like this paper we focus on topological data analysis (TDA), which is a broad field. What is interesting about join trees however, is that they provide for one of the most simple non-trivial approaches to topological data analysis. We use this as an excuse to motivate join trees through the role they play in this domain.

Two Setups in TDA

While there are several different approaches in topological data analysis, our first step is to turn our data into a continuous function. To be more precise, we associate a continuous function f to a real world phenomenon, that we seek to understand, and then we capture some data to which we associate a continuous function g approximating f. We start by loosely describing two different setups of a real world phenomenon, the type of data we capture, and how we obtain the continuous function g from this data.

Sensing continuous Functions

When we examine the physical environment around us, we may want to record our observations. Recording everything we observe is rarely possible however and therefore we often try to single out one elementary aspect and focus on this aspect alone. The most basic type of such a recording is that of a single quantity that we express as a real number $r \in \mathbb{R}$. To obtain such a value r, we often use some sensor that we place somewhere in our physical environment and then we read the sensor's value. In most situations the sensor will show a different value, when we place it somewhere else and even when we repeat the process and read the value from the sensor in the same location another time, the value may be different. In the latter situation the value is likely to be similar however and in the former situation the value is likely to be similar, if the new location is very close to the previous location. We assume that there is a certain subspace X of the environment around us within our reach and interest, where we can place the sensor. As a subspace of our environment, X inherits a topology and the above observations lead to the intuition, that there is a proper continuous function $f\colon X \to \mathbb{R}$ whose values we are reading of the sensor in proximity. This is also called a scalar field. Now we may never learn this function f, but we can interpolate between our samples from a sufficiently dense set of measuring locations, to define another continuous function $g\colon X \to \mathbb{R}$ that is close to f in the sense that $-\varepsilon \leq f(p) - g(p) \leq \varepsilon$ for all $p \in X$ and a real number $\varepsilon > 0$ of moderate size. More specifically we can make ε as small as we like by increasing both, the density of our samples and the accuracy of our sensor. We concede that the problem of approximating f is non-trivial and that interpolation is a vast field outside the scope of this document.

Point Clouds of Shapes

Imagine a subspace S of our environment, denoted by X, is densely covered with icing sugar. Further we assume we can see the icing sugar but not the space S itself. Now let $f\colon X \to \mathbb{R}$ be the function that assigns to each point in X it's distance to S and let $g\colon X \to \mathbb{R}$ assign to each point the distance to a nearest grain of icing sugar. Then S is the zero locus of f. Moreover g can be as close to f as we like, if we only spread the icing sugar densely enough. The icing sugar is also referred to as *point-cloud data for S*.

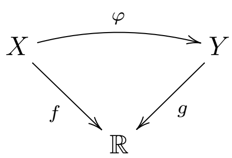
Distances on Functions

Above we discussed how we may associate a continuous function to a real world phenomenon. Now we explore what we can do with a continuous function or an approximation thereof and the conclusions we can draw from this about the original phenomenon. Let us assume we have two continuous functions $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ with approximations $f'\colon X \to \mathbb{R}$ respectively $g'\colon Y \to \mathbb{R}$.

Absolute Distance of Functions

We consider the following

(1) **Question.** Is there a homeomorphism $\varphi\colon X \to Y$ such that



commutes?

Example. We view a grayscale image as a function defined on a rectangle describing the luminance of each point. The following image shows the logo of the university of Bonn in grayscale.

And this image shows a warped version of the logo.

If interpret the two images as functions on a rectangular shape, we can get the second image from the first by precomposition with a homeomorphism φ describing the warp we used. Roughly, for two images, the answer to the above question is *yes*, if we can obtain one image from the other through a warp.

As we obtain an image from sampling the luminance with a sensor at each pixel, the previous example corresponds to the first setup above. Let $S \subset X$ and $S' \subset Y$ be subspaces of metric spaces X respectively Y and let f respectively g describe the distance to S respectively S' as in the second setup above. Now suppose $\varphi\colon X \to Y$ is an isometry with $\varphi(S) = S'$, then the above diagram commutes. In some sense this means, that if S and S' describe the same shape, then the answer to the above questions is *yes*.

The above question is very narrow as we ask for equality. In most situations occurring in nature the answer is very likely to be *no*. But we may still try to quantify how *distant an affirmative answer is*. To this end we start with specifying an $\varepsilon \geq 0$ and asking the following

(2) **Question.** Is there a homeomorphism $\varphi\colon X \to Y$ such that for all $p \in X$ the estimates $-\varepsilon \leq f(p) - g(\varphi(p)) \leq \varepsilon$ hold?

Next we may ask, how large ε needs to be in order for the answer to be *yes*. This is the motivation behind the following

Definition. Let $M(f, g)$ be the infimum of all $\varepsilon \geq 0$ such that the answer to the previous question is affirmative. We name this *the absolute distance of f and g*.

Remark. Though we won't need it we have the following equation for M. Let $\mathcal{H}$ be the set of homeomorphisms from X to Y, then

$$M(f,g) = \inf_{\varphi\in\mathcal{H}} \|f - g\circ\varphi\|_\infty.$$

Next we observe that M satisfies the triangle inequality.

Lemma (Triangle Inequality)**.** Let $h\colon Z \to \mathbb{R}$ be another continuous function, then $M(f,h) \le M(f,g) + M(g,h)$.

Now we recall that we started with two continuous functions f and g and each might describe the luminance of an image or the distance to a certain shape and this is all good, but we should not forget that in practice we may never learn f or g and we have to cope with their approximations f' respectively g'. In topological data analysis the keyword here is *stability* ever since Cohen-Steiner, Edelsbrunner, and Harer (2005).

(3) **Lemma** (Stability)**.** Suppose we have $\|f - f'\|_\infty = \varepsilon$, then $M(f,f') \le \varepsilon$.

Proof. Setting $\varphi = \mathrm{id}_X$ we get an affirmative answer to question 2.

The previous two lemmata have the following

Corollary. Suppose we have $\|f - f'\|_\infty \le \varepsilon \ge \|g - g'\|_\infty$,
then $|M(f',g') - M(f,g)| \le 2\varepsilon$.

In summary M defines an extended pseudometric on the class of continuous functions and the above corollary shows that by approximating the functions f and g, we can also approximate their absolute distance.

Relative Distance of Functions

We reconsider the first setup above where we take samples to approximate the continuous functions $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$. Now we imagine that the region X, where we sample f, is miles away from the region Y, where we sample g. Then it may be impractical to use the same sensor for both f and g. It would be more practical if someone who lived near X took the samples for f and someone else who lived near Y took the samples for g, each of them using their own sensor. Now suppose that any value read from the sensor used for sampling f always differed by roughly the same additive constant $r \in \mathbb{R}$ from the value that would have been read of the other sensor in the same location. The two sensors were so far apart however, that we could not measure this value r nor could we calibrate one sensor to match the other. To work around this dilemma we consider a slight modification of question 1.

Question. Is there a homeomorphism $\varphi\colon X \to Y$ and a real number r such that

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{R} & \xrightarrow[(r+\_)]{} & \mathbb{R} \end{array}$$

commutes? (In other words $r + f(p) = g(\varphi(p))$ for all $p \in X$.)

Again this is a very narrow question and in nature the answer is likely to be *no*. Nevertheless we may try to quantify how *distant an affirmative answer is.* To this end we specify $\varepsilon \geq 0$ and ask the following

Question. Is there a homeomorphism $\varphi\colon X \to Y$ and a real number r such that for all $p \in X$ the estimates $-\varepsilon \leq r + f(p) - g(\varphi(p)) \leq \varepsilon$ hold?

Now we minimize over all ε that provide an affirmative answer.

Definition. Let $\mu(f, g)$ be the infimum of all $\varepsilon \geq 0$ such that the answer to the previous question is affirmative. We name this *the relative distance of f and g.*

Remark. Though we won't need it, we have the following alternative description for $\mu(f, g)$ if $X \neq \emptyset$. Let $\mathcal{H}$ be the set of homeomorphisms from X to Y, then

$$\mu(f, g) = \frac{1}{2} \inf_{\varphi \in \mathcal{H}} \left(\sup_{p \in X} (f(p) - g(\varphi(p))) - \inf_{p \in X} (f(p) - g(\varphi(p))) \right).$$

Completely analogously to the above we have a triangular inequality.

Lemma (Triangle Inequality)**.** Let $h\colon Z \to \mathbb{R}$ be another continuous function, then $\mu(f, h) \leq \mu(f, g) + \mu(g, h)$.

Moreover we have the following relation to the absolute distance.

Lemma. We have $\mu(f, g) \leq M(f, g)$.

(4) **Corollary** (Stability)**.** Suppose we have $\|f - f'\|_\infty = \varepsilon$, then $\mu(f, f') \leq \varepsilon$.

Proof. This follows in conjunction with lemma 3.

The Triangle Inequality and Stability have the following

Corollary. Suppose we have $\|f - f'\|_\infty \leq \varepsilon \geq \|g - g'\|_\infty$,
then $|\mu(f', g') - \mu(f, g)| \leq 2\varepsilon$.

In summary μ defines an extended pseudometric on the class of continuous functions and the above corollary shows that by approximating the functions f and g, we can also approximate their relative distance.

Lower Bounds

While it may be quite hard to compute the two distances M and μ defined above we may try to compute lower bounds. Topological data analysis is a vast field and several solutions exist in this direction. Here we focus on two of them, namely the interleaving distance of join trees introduced by Morozov, Beketayev, and Weber (2013) and the interleaving distance of Reeb graphs by de Silva, Munch, and Patel (2016).

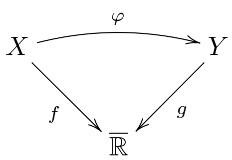
Join trees and Reeb graphs themselves are very similar. The join tree associated to a function f can be seen as a real-valued function itself whose fibers encode the connected components of the corresponding sublevel sets of f and the Reeb graph can be seen as a function whose fibers encode the connected components of the level sets. Join trees are much easier to work with however. While Morozov, Beketayev, and Weber (2013) defined their interleaving distance in an ad hoc manner, de Silva, Munch, and Patel (2016) introduce set-valued (pre)cosheaves first and then they define their interleaving distance. (Actually de Silva, Munch, and Patel (2016) introduce a second interleaving distance, that could be seen as a more ad hoc approach to the problem, but in the author's opinion[1] their first notion of an interleaving distance using the theory of precosheaves is more convenient for our considerations.)

Here is the main motivation behind this document. For a continuous function f Morozov, Beketayev, and Weber (2013) define the join tree of f as the Reeb graph of the epigraph associated to f. Later de Silva, Munch, and Patel (2016) introduced the *interleaving distance of Reeb graphs* and therefore there are now two different notions of an interleaving distance on join trees. For two continuous functions $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ we have the interleaving distance of join trees associated to f and g as defined by Morozov, Beketayev, and Weber (2013) and we have the interleaving distance of the Reeb graphs associated to the epigraph of f respectively g as defined by de Silva, Munch, and Patel (2016). We aim to show that the two distances are the same when X and Y are compact smooth manifolds.

It is not really essential and more of a personal preference that from this point onward we work with functions with values in the extended real line $\overline{\mathbb{R}} := [-\infty, \infty]$ and consider real-valued functions a subclass.

When working with different notions of an interleaving distance and in particular when comparing them, the use of some basic category theory seems very natural and simplifies several of our arguments. More specifically we will use functors and natural transformations on several occasions. Moreover we augment the class of $\overline{\mathbb{R}}$-valued continuous functions with the structure of a category, the *category of* $\overline{\mathbb{R}}$*-spaces* for short.

> **Definition.** For two continuous functions $f\colon X \to \overline{\mathbb{R}}$ and $g\colon Y \to \overline{\mathbb{R}}$, a *homomorphism* φ *from* f *to* g, also denoted by $\varphi\colon f \to g$, is a continuous map $\varphi\colon X \to Y$ such that the diagram



[1] the author of this document

commutes.

We define the composition of homomorphisms in the *category of* $\overline{\mathbb{R}}$*-spaces* by the composition of maps.

The *category of* $\mathbb{R}$*-spaces* is the full subcategory of all real-valued continuous functions.

Remark. With these definitions in place a reformulation of question 1 is, whether f and g are isomorphic as objects of the category of $\overline{\mathbb{R}}$-spaces.

Further References

Join trees have been harnessed for a similar purpose by Saikia, Seidel, and Weinkauf (2014). Further methods to compare Reeb graphs have been proposed by Bauer, Ge, and Wang (2014) and for the special case of functions on orientable surfaces by Di Fabio and Landi (2014). The two distances from (Bauer, Ge, and Wang 2014) and (de Silva, Munch, and Patel 2016) have been shown to be equivalent in some sense by Bauer, Munch, and Wang (2015). Moreover the work presented here borrows from (Bubenik, de Silva, and Scott 2014) on several aspects. There is much more prior art and the papers we mentioned merely represent our primary references. Most of the papers mentioned by de Silva, Munch, and Patel (2016) and Bubenik, de Silva, and Scott (2014) in their introductions are antecedents to our work as well. We presented some of the abstract ideas we use here in the form of a poster (Fluhr 2017). Little did we know that de Silva, Munch, and Stefanou (2017) were developing a very similar framework. Some differences between their approach and ours is that they work in a very general setup with an arbitrary metric space whereas we merely consider the real numbers as a base space. This made it feasible for us to treat the absolute and the relative interleaving distance in one go.

Reeb Graphs

In this section we introduce Reeb graphs.

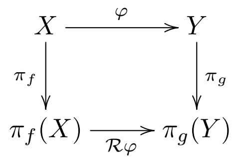
Definition (Reeb Graph)**.** Given a continuous function $f\colon X \to \overline{\mathbb{R}}$ and $x \in X$, let $\pi_f(x)$ be the connected component of x in $f^{-1}(f(x))$. In this way we obtain a function $\pi_f\colon X \to \pi_f(X) \subset 2^X$ and we endow $\pi_f(X)$ with the quotient topology[2]. By the universal property of the quotient space there is a unique continuous function $\mathcal{R}f\colon \pi_f(X) \to \overline{\mathbb{R}}$ such that

$$\begin{array}{ccc} X & \xrightarrow{\pi_f} & \pi_f(X) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \mathcal{R}f} \\ & & \overline{\mathbb{R}} \end{array}$$

commutes, i.e. $\mathcal{R}f \circ \pi_f = f$.

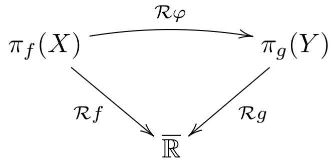
[2] see for example (Bredon 1993, definition I.13.1)

For another continuous function $g\colon Y \to \overline{\mathbb{R}}$ and a homomorphism $\varphi\colon f \to g$ we may use the universal property of π_f again, to obtain a unique continuous map $\mathcal{R}\varphi\colon \pi_f(X) \to \pi_g(Y)$ such that



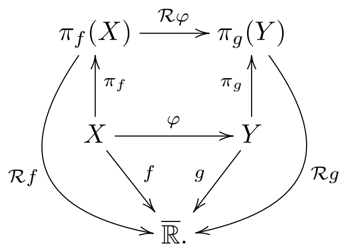
commutes, i.e. $\mathcal{R}\varphi \circ \pi_f = \pi_g \circ \varphi$.

Lemma. Let $f\colon X \to \overline{\mathbb{R}}$ and $g\colon Y \to \overline{\mathbb{R}}$ be continuous functions and let $\varphi\colon f \to g$ be a homomorphism, then the diagram



commutes. Or in other words $\mathcal{R}\varphi$ is a homomorphism from $\mathcal{R}f$ to $\mathcal{R}g$ in the category of $\overline{\mathbb{R}}$-spaces.

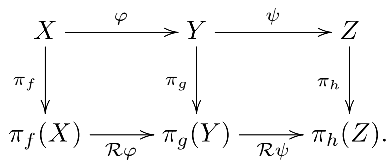
Proof. We consider the diagram



In this diagram the three inner triangles and the square commute. Further π_f and π_g are surjective and thus the outer triangle commutes as well.

Lemma. Let $f\colon X \to \overline{\mathbb{R}}$, $g\colon Y \to \overline{\mathbb{R}}$, and $h\colon Z \to \overline{\mathbb{R}}$ be continuous functions and let $\varphi\colon f \to g$ and $\psi\colon g \to h$ be two homomorphisms, then $\mathcal{R}(\psi \circ \varphi) = \mathcal{R}\psi \circ \mathcal{R}\varphi$.

Proof. We consider the diagram



By definition both inner squares commute, hence the outer square commutes as well. Now it follows from the uniqueness part of the universal property of π_f that $\mathcal{R}(\psi \circ \varphi) = \mathcal{R}\psi \circ \mathcal{R}\varphi$.

The previous two lemmata imply that $\mathcal{R}$ is an endofunctor on the category of $\overline{\mathbb{R}}$-spaces. Later we will define an interleaving distance on Reeb graphs, but first we will introduce join trees and their interleavings. Join trees are easier to understand and may provide us with some intuition for understanding the more sophisticated interleavings of Reeb graphs.

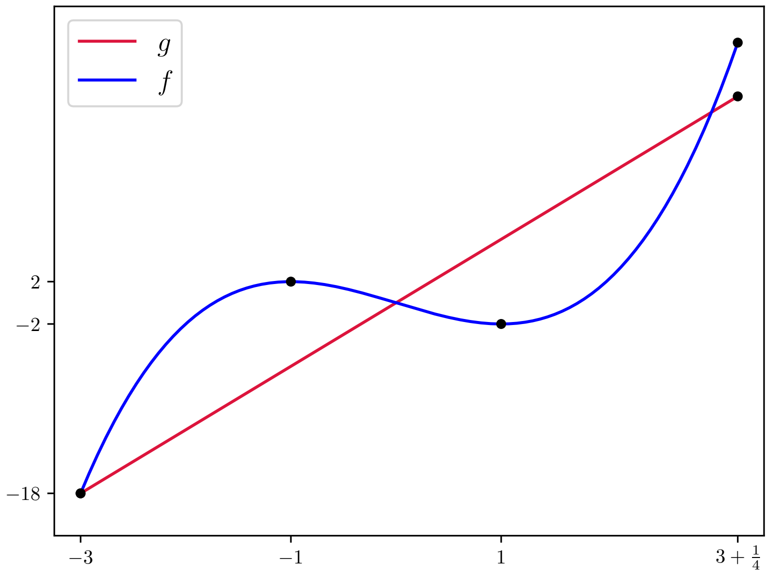
Join Trees

In this section we define join trees and their interleaving distance due to Morozov, Beketayev, and Weber (2013). We start with an auxiliary

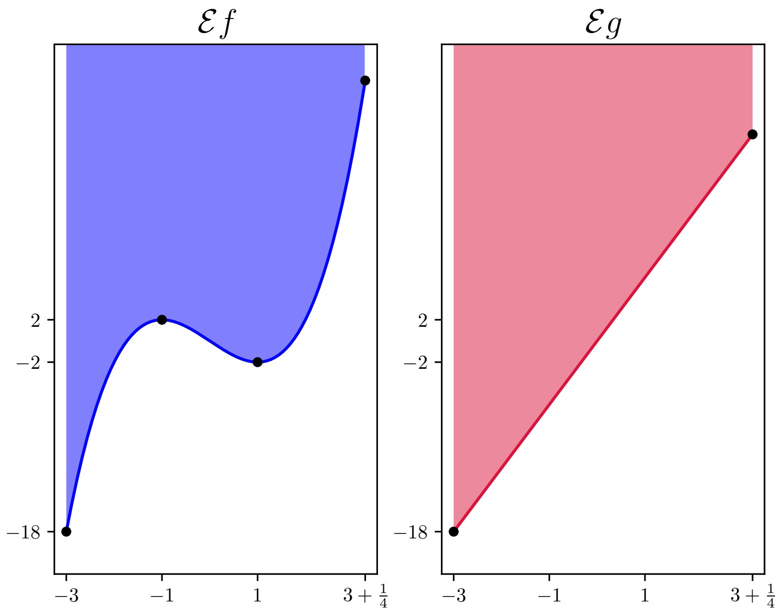
> **Definition** (Epigraph)**.** Let $f\colon X \to \overline{\mathbb{R}}$ be a continuous map, it's *epigraph* is $\operatorname{epi} f := \{(x, y) \in X \times \overline{\mathbb{R}} \mid y \geq f(x)\}$.
> Further we define
> $\mathcal{E}f\colon \operatorname{epi} f \to \overline{\mathbb{R}}, (x, y) \mapsto y$ and
> $\kappa_f\colon X \to \operatorname{epi} f, x \mapsto (x, f(x))$.
> For $a \geq 0$ we set $i_f^a\colon \operatorname{epi} f \to \operatorname{epi} f, (x, y) \mapsto (x, y + a)$.

The epigraph and the Reeb graphs from the previous section allow for a very brief definition of join trees.

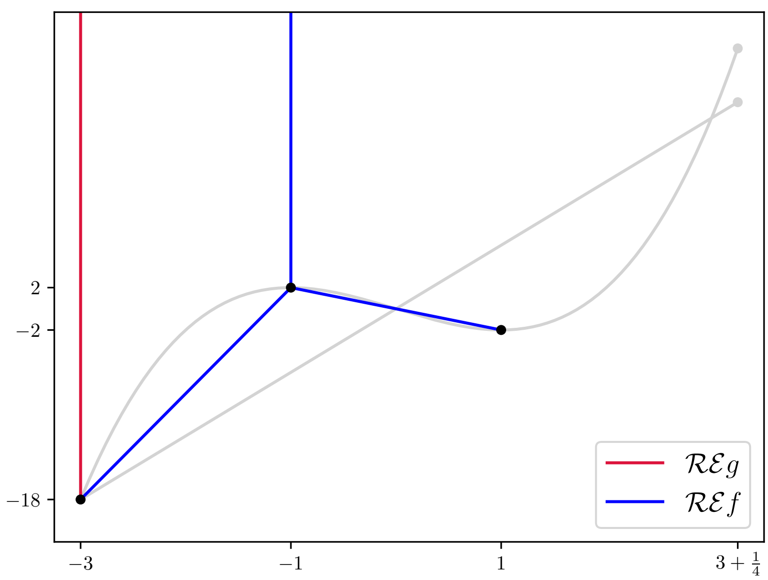
> **Definition** (Join Trees)**.** Let $f\colon X \to \mathbb{R}$ be a continuous map, it's *join tree* is $\mathcal{R}\mathcal{E}f$.
>
> *Example.* Let $f\colon \left[-3, 3+\frac{1}{4}\right] \to \mathbb{R}, x \mapsto x^3 - 3x$ and $g\colon \left[-3, 3+\frac{1}{4}\right] \to \mathbb{R}, x \mapsto 6x$. The following image shows the graphs of f and g.



The next image shows the corresponding epigraphs.



And a visualization of their join trees, i.e. the Reeb graphs of their epigraphs, is shown in the image below.



Before we get to interleavings we make some auxiliary definitions.

Definition. We set $D^{\perp} := \{(a,b) \in (-\infty,\infty] \times (-\infty,\infty] \mid a+b \geq 0\}$.

Component-wise addition yields the structure of a monoid on $D^{\perp}$. Next we define two weightings on $D^{\perp}$.

Definition. We set $\epsilon'\colon D^{\perp} \to [0,\infty], (a,b) \mapsto \frac{1}{2}(a+b)$ and $\epsilon''\colon D^{\perp} \to [0,\infty], (a,b) \mapsto \max\{a,b\}$.

We note that ϵ' is additive and that ϵ'' is subadditive. Now we define interleavings of join trees. To this end let $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ be continuous functions.

Definition (Interleavings of Join Trees)**.** For $(a,b) \in D^{\perp}$ an (a,b)-*interleaving of* $\mathcal{RE}f$ *and* $\mathcal{RE}g$ is a pair of homomorphisms $\varphi\colon b+\mathcal{RE}f \to \mathcal{RE}g$ and $\psi\colon a+\mathcal{RE}g \to \mathcal{RE}f$ such that the diagrams

$$\begin{array}{ccc} a+b+\mathcal{RE}f & \xrightarrow{(\mathcal{R}\circ i^{a+b})_f} & \mathcal{RE}f \\ & {\scriptstyle\varphi}\searrow \quad \nearrow{\scriptstyle\psi} & \\ & a+\mathcal{RE}g & \end{array}$$

and

$$\begin{array}{ccc} a+b+\mathcal{RE}g & \xrightarrow{(\mathcal{R}\circ i^{a+b})_g} & \mathcal{RE}g \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\varphi} & \\ & b+\mathcal{RE}f & \end{array}$$

commute.

For $\varepsilon \geq 0$ an ε-*interleaving of* $\mathcal{RE}f$ *and* $\mathcal{RE}g$ is an $(\varepsilon,\varepsilon)$-interleaving.

Example. We consider the join trees from the previous example. We set $\varphi\colon \pi_{\mathcal{E}f}(\operatorname{epi} f) \to \pi_{\mathcal{E}g}(\operatorname{epi} g), \pi_{\mathcal{E}f}((x,y)) \mapsto \pi_{\mathcal{E}g}((-3,y+2))$ and $\psi\colon \pi_{\mathcal{E}g}(\operatorname{epi} g) \to \pi_{\mathcal{E}f}(\operatorname{epi} f), \pi_{\mathcal{E}g}((x,y)) \mapsto \pi_{\mathcal{E}f}((-3,y+2))$, then φ and ψ form a $(2,2)$-interleaving of $\mathcal{RE}f$ and $\mathcal{RE}g$.

With interleavings defined we can define the interleaving distances.

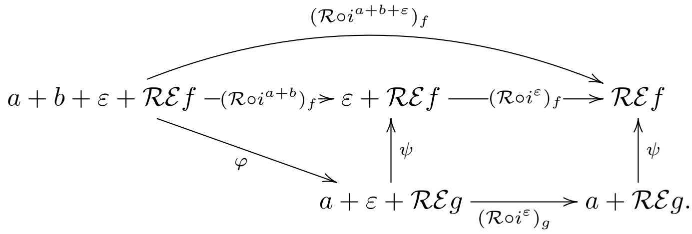
Definition (Interleaving Distances of Join Trees)**.** Let $\mathcal{I}$ be the set of all $\mathbf{a} \in D^{\perp}$ such that there is an $\mathbf{a}$-interleaving of $\mathcal{RE}f$ and $\mathcal{RE}g$. Then we set $M_J(\mathcal{RE}f, \mathcal{RE}g) := \inf \epsilon''(\mathcal{I})$ and $\mu_J(\mathcal{RE}f, \mathcal{RE}g) := \inf \epsilon'(\mathcal{I})$.

We name $M_J(\mathcal{RE}f, \mathcal{RE}g)$ the *absolute interleaving distance of* $\mathcal{RE}f$ *and* $\mathcal{RE}g$ and $\mu_J(\mathcal{RE}f, \mathcal{RE}g)$ the *relative interleaving distance of* $\mathcal{RE}f$ *and* $\mathcal{RE}g$.

We note that these definitions of an interleaving distance are different from the one by Morozov, Beketayev, and Weber (2013). However the subsequent corollary shows that our absolute interleaving distance is equal to the distance by Morozov, Beketayev, and Weber (2013). The reason we gave a different definition is that now we can phrase the computation of the absolute and the relative interleaving distance each as an optimization problem over the same domain.

Lemma (Monotonicity)**.** For $(a,b) \in D^{\perp}$ let $\varphi\colon b + \mathcal{RE}f \to \mathcal{RE}g$ and $\psi\colon a + \mathcal{RE}g \to \mathcal{RE}f$ be an (a,b)-interleaving of $\mathcal{RE}f$ and $\mathcal{RE}g$. Moreover let $\varepsilon \geq 0$. Then $(\mathcal{R} \circ i^{\varepsilon})_g \circ \varphi$ and ψ yield an $(a, b+\varepsilon)$-interleaving. Completely analogously φ and $(\mathcal{R} \circ i^{\varepsilon})_f \circ \psi$ form an $(a+\varepsilon, b)$-interleaving.

Proof. We consider the diagram



Since all the inner triangles and the square commute in the above diagram, the outer triangle commutes as well. The other triangle, for $(\mathcal{R}\circ i^{\varepsilon})_g \circ \varphi$ and ψ to be an interleaving, also commutes, because we have $(\mathcal{R}\circ i^{\varepsilon})_g \circ (\mathcal{R}\circ i^{a+b})_g = (\mathcal{R}\circ i^{a+b+\varepsilon})_g$.

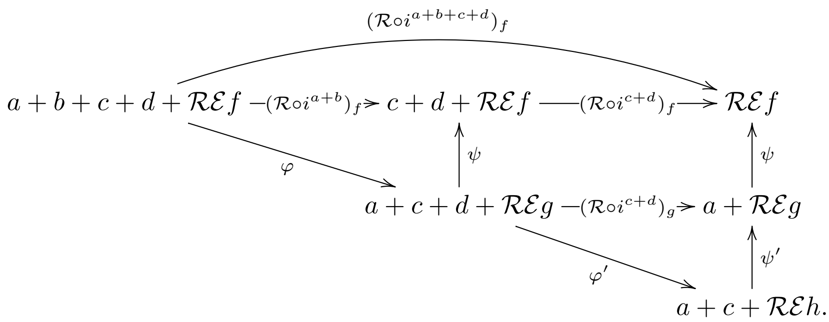
Corollary. The absolute interleaving distance $M_J(\mathcal{RE}f, \mathcal{RE}g)$ is the infimum of all $\varepsilon \geq 0$ such that there is an ε-interleaving of $\mathcal{RE}f$ and $\mathcal{RE}g$.

Proof. Let d be the infimum of all $\varepsilon \geq 0$ such that there is an ε-interleaving of $\mathcal{RE}f$ and $\mathcal{RE}g$. Then the inequality $d \leq M_J(\mathcal{RE}f, \mathcal{RE}g)$ is clear. For the other inequality suppose we have an (a,b)-interleaving of $\mathcal{RE}f$ and $\mathcal{RE}g$. Then the previous lemma implies that we also have an $(\epsilon''(a,b))$-interleaving of $\mathcal{RE}f$ and $\mathcal{RE}g$.

Next we underpin the naming conventions from the previous definition by proving a type of triangle inequality for each interleaving distance. To this end let $h\colon Z \to \mathbb{R}$ be yet another continuous function.

Lemma. Let $(a,b), (c,d) \in D^{\perp}$. Further let $\varphi\colon b + \mathcal{RE}f \to \mathcal{RE}g$ and $\psi\colon a + \mathcal{RE}g \to \mathcal{RE}f$ be an (a,b)-interleaving of $\mathcal{RE}f$ and $\mathcal{RE}g$ and let $\varphi'\colon d + \mathcal{RE}g \to \mathcal{RE}h$ and $\psi'\colon c + \mathcal{RE}h \to \mathcal{RE}g$ be a (c,d)-interleaving of $\mathcal{RE}g$ and $\mathcal{RE}h$. Then $\varphi' \circ \varphi$ and $\psi \circ \psi'$ yield an $(a+c, b+d)$-interleaving of $\mathcal{RE}f$ and $\mathcal{RE}h$.

Proof. We consider the diagram



The upper triangle commutes since $i_f^{a+b+c+d} = i_f^{c+d} \circ i_f^{a+b}$ and since $\mathcal{R}$ is a functor. The other two inner triangles and the square commute by definition and thus the outer triangle commutes as well. The proof that the other triangle, for $\varphi' \circ \varphi$ and $\psi \circ \psi'$ to be an interleaving, commutes, is completely analogous.

Corollary (Triangle Inequality)**.** We have

$$M_J(\mathcal{R}\mathcal{E}f, \mathcal{R}\mathcal{E}h) \leq M_J(\mathcal{R}\mathcal{E}f, \mathcal{R}\mathcal{E}g) + M_J(\mathcal{R}\mathcal{E}g, \mathcal{R}\mathcal{E}h)$$

and

$$\mu_J(\mathcal{R}\mathcal{E}f, \mathcal{R}\mathcal{E}h) \leq \mu_J(\mathcal{R}\mathcal{E}f, \mathcal{R}\mathcal{E}g) + \mu_J(\mathcal{R}\mathcal{E}g, \mathcal{R}\mathcal{E}h).$$

Next we show that the interleaving distances provide lower bounds for the corresponding distances of functions as promised.

Proposition. Suppose we have a homeomorphism $\varphi\colon X \to Y$, $\varepsilon \geq 0$, and $r \in \mathbb{R}$ such that $-\varepsilon \leq r + f(p) - g(\varphi(p)) \leq \varepsilon$ for all $p \in X$. Let $\varphi'\colon \operatorname{epi} f \to \operatorname{epi} g, (p, u) \mapsto (\varphi(p), u + r + \varepsilon)$ and $\psi'\colon \operatorname{epi} g \to \operatorname{epi} f, (q, v) \mapsto (\varphi^{-1}(q), v - r + \varepsilon)$. Then $\mathcal{R}\varphi'$ and $\mathcal{R}\psi'$ are an $(-r + \varepsilon, r + \varepsilon)$-interleaving of $\mathcal{R}\mathcal{E}f$ and $\mathcal{R}\mathcal{E}g$.

Corollary. We have $M_J(\mathcal{R}\mathcal{E}f, \mathcal{R}\mathcal{E}g) \leq M(f, g)$ and $\mu_J(\mathcal{R}\mathcal{E}f, \mathcal{R}\mathcal{E}g) \leq \mu(f, g)$.

This corollary concludes our treatment of join trees by themselves. The next step concerning join trees would be to describe practical algorithms to compute them and to compute each of their interleaving distances, but this is outside the scope of this document. Next we consider other more elaborate approaches to finding lower bounds to the absolute and relative distance of functions. Nevertheless the main stream of arguments will always be very similar to the one in this section. We will just use more auxiliary results from domains other than topological data analysis. After we covered Reeb graphs we will even describe a more general framework that encompasses both join trees and Reeb graphs, so we could actually abandon the results from this section. At the expense of some repetition we chose not to do this because this section contains everything that follows in a nutshell.

Interlude on Precosheaves

In this section we develop the theory of precosheaves to the extend needed for the interleaving distance of Reeb graphs by de Silva, Munch, and Patel (2016) and subsequent sections. Before we get to precosheaves we start with some point-set topological definitions.

Definition (Intersection-Base)**.** Let X be a set. An *intersection-base on* X is a collection $\mathcal{B}$ of subsets of X that covers X and such that for any $U, V \in \mathcal{B}$ we have $U \cap V \in \mathcal{B}$.

Example. Let X be a topological space, then the topology on X is an intersection-base on X.

The set of open intervals of $\mathbb{R}$ is an intersection-base on $\mathbb{R}$.

Definition (Intersection-based space)**.** An *intersection-based space* is a set X together with an intersection-base $\mathcal{B}$ on X.

For any $U \in \mathcal{B}$ we say that U is a *distinguished open subset of* X.

Similar to topological spaces we can equip subsets of an intersection-based space with an induced intersection-base.

Lemma. Let X be a set and $\mathcal{B}$ be an intersection base on X. For a subset $Y \subseteq X$ the family $\{Y \cap U\}_{u \in \mathcal{B}}$ is an intersection-base on Y.

Definition. In the context of the previous lemma we say Y *is a subspace of* X.

In addition to subspaces we also define products.

Definition (Product)**.** Let X and Y be intersection-based spaces with intersection-bases $\mathcal{B}_X$ respectively $\mathcal{B}_Y$. We augment $X \times Y$ with the intersection-base $\mathcal{B}_X \times \mathcal{B}_Y$ and name this space the *product* $X \times Y$ *of* X *and* Y.

Next we define continuous maps between intersection-based spaces.

Definition. For two intersection-based spaces X and Y a map of sets $\varphi\colon X \to Y$ is said to be *continuous* if for any distinguished open subset $V \subseteq Y$ it's preimage $\varphi^{-1}(V)$ is a distinguished open subset of X.

Remark. The previous definition augments the class of intersection-based spaces with the structure of a category that contains the category of topological spaces as a full subcategory.

Example. Suppose Y is a subspace of the intersection-based space X, then the inclusion of Y into X is continuous.

Lemma. Let X be a topological space, let Y be a set, and let $\mathcal{B}$ be an intersection-base on Y. Then a map of sets $\varphi\colon X \to Y$ is continuous as a map between intersection-based spaces if and only if it is continuous with respect to the topology on Y induced by $\mathcal{B}$.

Remark. The previous lemma implies that the category of topological spaces is a coreflective subcategory of the category of intersection-based spaces.

As we can see intersection-based spaces and topological spaces are very similar and so far intersection-based spaces didn't provide us with anything new. It is the following definition where intersection-based spaces provide something that their corresponding topological spaces my not provide.

Definition. Let X and Y be intersection-based spaces with intersection-bases $\mathcal{B}_X$ respectively $\mathcal{B}_Y$. A continuous map $\varphi\colon X \to Y$ is *Galois* if for all $U \in \mathcal{B}_X$ the set $\{V \in \mathcal{B}_Y \mid U \subseteq \varphi^{-1}(V)\}$ has a minimum which we denote by $\varphi^{+1}(U)$.

The reason we name such a map φ Galois is that φ^{+1} and φ^{-1} then form a monotone Galois connection $\varphi^{+1} \dashv \varphi^{-1}$.

(5) **Lemma.** Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be continuous and Galois, then $\psi \circ \varphi$ is Galois and $(\psi \circ \varphi)^{+1} = \psi^{+1} \circ \varphi^{+1}$.

Proof. This follows from a general statement about Galois connections, see for example (Erné et al. 1993, proposition 2).

These are all the point-set topological notions we need, so we can start with precosheaves.

Definition (Set-valued Precosheaves)**.** Let X be an intersection-based space with intersection base $\mathcal{B}$. A set-valued precosheaf F on X is a functor from $\mathcal{B}$, partially ordered by inclusions, to the category of sets.

The reader may already guess that for an intersection-based space X we will also augment the class of set-valued precosheaves on X with the structure of a category.

Definition. Let F and G be precosheaves on some intersection-based space X. A homomorphism $\alpha\colon F \to G$ is a natural transformation from F to G.

Composition of homomorphisms is given by composition of natural transformations.

Now let $\varphi\colon X \to Y$ be a continuous map between intersection-based spaces. We associate to φ a functor from the category of set-valued precosheaves on X to the category of precosheaves on Y.

Definition (Pushforward)**.** For a precosheaf F on X we set $\varphi_* F := F \circ \varphi^{-1}$ and for a homomorphism $\alpha\colon F \to G$ we set $\varphi_* \alpha := \alpha \circ \varphi^{-1}$. Here we view φ^{-1} as a monotone map from the intersection-base of Y to the intersection-base of X. We name $\varphi_* F$ the *pushforward of F by φ.*

We note that φ_* is a functor from the category of precosheaves on X to the category of precosheaves on Y. Now let $\psi\colon Y \to Z$ be another continuous map between intersection-based spaces.

Lemma. We have $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

Now suppose φ and ψ are Galois. We describe a functor in the other direction.

Definition (Pullback)**.** For a precosheaf G on Y we set $\varphi_p G := G \circ \varphi^{+1}$ and for a homomorphism $\alpha\colon F \to G$ we set $\varphi_p \alpha := \alpha \circ \varphi^{+1}$. We name $\varphi_p G$ the *pullback of G by φ.*

We note that φ_p is a functor from the category of precosheaves on Y to the category of precosheaves on X.

Lemma. We have $(\psi \circ \varphi)_p = \varphi_p \circ \psi_p$.

Proof. This follows from lemma 5.

Definition. For all distinguished open subsets $U \subseteq X$ we have $U \subseteq (\varphi^{+1} \circ \varphi^{-1})(U)$ and this yields a natural homomorphism $\eta_F^{\varphi} : F \to \varphi_p \varphi_* F$ for any precosheaf F on X.

Conversely we have $(\varphi^{-1} \circ \varphi^{+1})(V) \subseteq V$ for all distinguished open subsets $V \subseteq Y$. This induces a natural homomorphism $\varepsilon_G^{\varphi} : \varphi_* \varphi_p G \to G$ for any precosheaf G on Y.

If φ is not Galois then a functor like φ_p still exists, see for example (Stacks Project Authors 2017, tag 008C), but it is not as easy to define and not as easy to work with. In particular the previous lemma would not hold in this form and such compositions would invoke some subtleties. This is the reason we introduced intersection-based spaces in the first place.

Definition (Restriction)**.** For a subspace Y of an intersection-based space X whose inclusion is Galois and a precosheaf F on X we denote the pullback of F by the inclusion by $F|_Y$.

Monoidal Posets for 1D-Interleavings

Before we defined interleavings of join trees we introduced the poset $D^\perp$ and the two weightings ϵ' and ϵ'' on $D^\perp$. Morozov, Beketayev, and Weber (2013) used a more ad hoc approach and we justified our approach by also introducing the relative interleaving distance. With Reeb graphs we will do something similar and we introduce more formalism than de Silva, Munch, and Patel (2016). The reason for this is two fold. Starting from the approach by de Silva, Munch, and Patel (2016) the author added one layer of formalism to define the relative interleaving distance and then another layer of formalism, which may help with the computation of these interleaving distances. The nice thing about join trees is that we can get both benefits with just one additional layer of formalism whereas here we need two layers, so that a dedicated section is required. In other words we beg the reader to bear with us for one more section until we get to the absolute and relative interleaving distances of Reeb graphs.

Definition. Let $D := \{(a, b) \in [-\infty, \infty) \times (-\infty, \infty] \mid a \leq b\}$ then D is a monoid by component-wise addition. Moreover we set $\mathbf{o} := (0, 0)$ and we define a partial order $\preceq$ on D by $(x, y) \preceq (x', y') :\Leftrightarrow x \geq x' \wedge y \leq y'$.

A set augmented with a partial ordering and a monotone monoid structure like D, we call a *monoidal poset*. Moreover $D \times D$ is a monoidal poset with the product ordering and component-wise addition. With some abuse of notation we write $(a, b; c, d)$ for points $((a, b), (c, d)) \in D \times D$. Next we define three monoidal sub-posets of $D \times D$.

Definition. We set
$\mathcal{D} := \{(a, b; c, d) \in D \times D \mid a + c \leq 0 \leq b + d\}$,
$\triangledown := \{(a, b; c, d) \in \mathcal{D} \mid c + b = 0 = a + d\}$, and
$\blacktriangledown := \{(a, b; c, d) \in \triangledown \mid a + b = 0\}$.

So now we have the chain of monoidal posets $\blacktriangledown \subset \triangledown \subset \mathcal{D}$. For each of the two inclusions we aim to describe a monotone subadditive map in the other direction.

Definition. For $(a, b; c, d) \in \mathcal{D}$ we set $\delta((a, b; c, d)) := (-d', b'; -b', d')$ where $b' = \max\{-c, b\}$ and $d' = \max\{-a, d\}$. And for $(a, b; c, d) \in \triangledown$ we set $\gamma((a, b; c, d)) := (-\varepsilon, \varepsilon; -\varepsilon, \varepsilon)$ where $\varepsilon = \max\{b, d\}$.

Lemma. For all $\mathbf{A} \in \mathcal{D}$ and $\mathbf{B} \in \triangledown$ with $\mathbf{A} \preceq \mathbf{B}$ we have $\delta(\mathbf{A}) \preceq \mathbf{B}$.

Similarly we have $\gamma(\mathbf{A}) \preceq \mathbf{B}$ for all $\mathbf{A} \in \triangledown$ and $\mathbf{B} \in \blacktriangledown$ with $\mathbf{A} \preceq \mathbf{B}$.

In other words δ and γ are lower adjoints to the corresponding inclusions in the sense of monotone Galois connections. Now we will first describe a weighting on $\triangledown$ and then we will harness these two maps to describe two weightings on $\mathcal{D}$.

Definition. We set $\epsilon \colon \triangledown \to [0, \infty], (a, b; c, d) \mapsto \frac{1}{2}(b + d)$.

We note that ϵ is monotone and additive. Moreover the restriction $\epsilon|_{\blacktriangledown}$ is an isomorphism of monoidal posets. The two different weightings on $\mathcal{D}$ are now $\epsilon \circ \gamma \circ \delta$ and $\epsilon \circ \delta$. Next we provide explicit formulas for these two weightings.

Lemma. For all $(a, b; c, d) \in \mathcal{D}$ we have

$$(\epsilon \circ \gamma \circ \delta)((a, b; c, d)) = \max\{-a, b, -c, d\}$$

and

$$(\epsilon \circ \delta)((a, b; c, d)) = \frac{1}{2}(\max\{-c, b\} + \max\{-a, d\}).$$

Interleaving Reeb Graphs

In this section we define the interleaving distance of Reeb graphs due to de Silva, Munch, and Patel (2016). Strictly speaking it is somewhat misleading to name this the interleaving distance of Reeb graphs because we are not comparing the Reeb graphs directly, instead we compare what we name the Reeb precosheaf[3]. de Silva, Munch, and Patel (2016) discuss very thoroughly, why we may do this and later they also translate this into a more geometric comparison of Reeb graphs. We define this Reeb precosheaf somewhat differently from the way it is defined in this article. Using the notion of an intersection-based space we may say that de Silva, Munch, and Patel (2016) define the Reeb precosheaf as a precosheaf on the intersection-based space, that

[3] de Silva, Munch, and Patel (2016) actually name it the Reeb cosheaf, because in their setup, using locally connected spaces, it always is a cosheaf, a precosheaf with further properties.

has the real numbers as an underlying set and the open intervals as distinguished open subsets. We will use a different space, but all of the distinguished open subsets of this space, except for three, can be identified with an open interval. Now because precosheaves are just functors on the distinguished open subsets, this is pretty much the same data. And the reason we can't identify all distinguished open subsets with an interval is that we work with $\overline{\mathbb{R}}$ instead of $\mathbb{R}$, in some sense $\overline{\mathbb{R}}$ has three more open intervals than $\mathbb{R}$. Now we begin with defining this intersection-based space.

Definition. We define $\overline{\mathbb{R}}_\infty$ to be the space that has $\overline{\mathbb{R}}$ as an underlying set and $\{\overline{\mathbb{R}}\} \cup \{(a, \infty] \mid a \in \overline{\mathbb{R}}\}$ as it's intersection-base.

Similarly we define $\overline{\mathbb{R}}_{-\infty}$ to be the space with the same underlying set and $\{\overline{\mathbb{R}}\} \cup \{[-\infty, b) \mid b \in \overline{\mathbb{R}}\}$ as it's intersection-base.

Finally we set $\overline{\mathbb{E}} := \overline{\mathbb{R}}_\infty \times \overline{\mathbb{R}}_{-\infty}$ and $\overline{D} := \{(x, y) \in \overline{\mathbb{E}} \mid x \leq y\}$.

We note that any functor from the category of topological spaces to the category of sets defines a precosheaf on any topological space X, since we may equip any open subset $U \subseteq X$ with the subspace topology and since inclusions of subsets are continuous with respect to this topology. In the following definition we define a functor on the category of topological spaces and in this way also a precosheaf on any topological space.

Definition. For topological space X we define $\Lambda(X)$ to be the set of connected components of X.

For a continuous map $\varphi\colon X \to Y$ we define $\Lambda(\varphi)$ to be the induced map from $\Lambda(X)$ to $\Lambda(Y)$.

Now let $f\colon X \to \overline{\mathbb{R}}$ and $g\colon Y \to \overline{\mathbb{R}}$ be continuous functions, let $\varphi\colon f \to g$ be a homomorphism in the category of $\overline{\mathbb{R}}$-spaces, and let $\Delta\colon \overline{\mathbb{R}} \to \overline{D}, t \mapsto (t, t)$. For a distinguished open subset $U \subseteq \overline{D}$ let $\varphi|_{(\Delta\circ f)^{-1}(U)}\colon (\Delta \circ f)^{-1}(U) \to (\Delta \circ g)^{-1}(U)$ be the restriction of φ to $(\Delta \circ f)^{-1}(U)$.

Definition (Reeb Precosheaf)**.** We set $\mathcal{C}f := (\Delta \circ f)_*\Lambda$ and name this the *Reeb precosheaf of f*.

Moreover we define a homomorphism $\mathcal{C}\varphi$ from $\mathcal{C}f$ to $\mathcal{C}g$. For a distinguished open subset $U \subseteq \overline{D}$ we set $(\mathcal{C}\varphi)_U := \Lambda\big(\varphi|_{(\Delta\circ f)^{-1}(U)}\big)$.

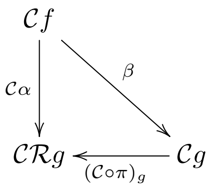
With these definitions $\mathcal{C}$ defines a functor from the category of $\overline{\mathbb{R}}$-spaces to the category of set-valued precosheaves on $\overline{D}$. Now assume that g is a constructible $\overline{\mathbb{R}}$-space[4]. We discuss the relationship of the Reeb graph $\mathcal{R}g$ and the Reeb precosheaf $\mathcal{C}g$.

(6) **Lemma.** The homomorphism $(\mathcal{C} \circ \pi)_g$ from $\mathcal{C}g$ to $\mathcal{C}\mathcal{R}g$ is an isomorphism of precosheaves.

The previous lemma implies that the isomorphism class of $\mathcal{C}g$ is determined by $\mathcal{R}g$. Next we will see that the converse is also true.

[4]see the first appendix for our definition of a constructible $\overline{\mathbb{R}}$-space

(7) **Lemma.** Let $\beta\colon \mathcal{C}f \to \mathcal{C}g$ be a homomorphism of precosheaves, then there is a unique homomorphism $\alpha\colon f \to \mathcal{R}g$ of $\overline{\mathbb{R}}$-spaces such that the diagram



commutes.

Remark. One can rephrase the previous lemma in the terminology of category theory in the following way. The homomorphism $(\mathcal{C}\circ\pi)_g^{-1}$ yields a representation of the contravariant functor $\mathrm{Hom}(\mathcal{C}\_, \mathcal{C}g)$. A vast generalization of this statement has been provided by Funk (1995).

The previous two lemmata have the following

Corollary. The map

$$\mathcal{C}(\_)\colon \mathrm{Hom}(f, \mathcal{R}g) \to \mathrm{Hom}(\mathcal{C}f, \mathcal{C}\mathcal{R}g), \alpha \mapsto \mathcal{C}\alpha$$

is a bijection.

Proof. First we note that the post-composition $(\mathcal{C}\circ\pi)_g^{-1}\circ(\_)$ yields a bijection from $\mathrm{Hom}(\mathcal{C}f, \mathcal{C}\mathcal{R}g)$ to $\mathrm{Hom}(\mathcal{C}f, \mathcal{C}g)$. Now the previous lemma states that the composition $(\mathcal{C}\circ\pi)_g^{-1}\circ\mathcal{C}(\_)$ is a bijection as well. Thus also $\mathcal{C}(\_)$ is a bijection.

(8) **Corollary.** The functor $\mathcal{C}$ is full and faithful on the full subcategory of Reeb graphs of constructible $\overline{\mathbb{R}}$-spaces.

The previous corollary implies that the isomorphism class of $\mathcal{R}g$ is determined by the isomorphism class of $\mathcal{C}\mathcal{R}g$ and this class in turn is determined by $\mathcal{C}g$, hence $\mathcal{C}g$ determines the isomorphism class of $\mathcal{R}g$.

In summary we may associate to any $\overline{\mathbb{R}}$-space a set-valued precosheaf, the Reeb precosheaf. In the following subsection we define two interleaving distances for set-valued precosheaves on $\overline{D}$. And for constructible $\overline{\mathbb{R}}$-spaces we may also refer to the interleaving distances of the Reeb precosheaves as interleaving distances of Reeb graphs. This is justified by the fact that de Silva, Munch, and Patel (2016) also define an interleaving distance of Reeb graphs, that is equal to the interleaving distance of the Reeb precosheaves for constructible $\overline{\mathbb{R}}$-spaces.

Interleaving Precosheaves on $\overline{D}$

In the interlude on precosheaves we described how continuous and Galois maps between intersection-based spaces yield functors between the corresponding categories of precosheaves. We start by describing some self-maps on $\overline{D}$ to yield us endofunctors on the category of precosheaves on $\overline{D}$.

Definition. For $a \in \mathbb{R}$ we define $s^a \colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}, t \mapsto t + a$ and we set $s^{\pm\infty} \colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}, t \mapsto \pm\infty$.

We note that s^a is continuous and Galois on $\overline{\mathbb{R}}_\infty$ for $-\infty \leq a < \infty$ and on $\overline{\mathbb{R}}_{-\infty}$ for $-\infty < a \leq \infty$. Thus the following definition yields a continuous and Galois self-map on $\overline{\mathbb{E}}$ for all $(a, b) \in D$.

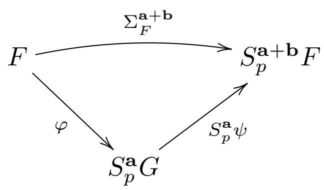
Definition. For $(a, b) \in D$ we set $S^{(a,b)} := (s^a \circ \pi^1) \times (s^b \circ \pi^2)$.

More explicitly we have $S^{(a,b)}(x, y) = (s^a(x), s^b(y))$ for all $(a, b) \in D$ and $x, y \in \overline{\mathbb{R}}$. For all $\mathbf{a} \in D$ we have $S^{\mathbf{a}}(\overline{D}) \subseteq \overline{D}$, hence $S^{\mathbf{a}}$ also defines a continuous and Galois self-map on $\overline{D}$. Now let F be a set-valued precosheaf on $\overline{D}$. Using the definitions from the interlude on precosheaves we get the precosheaf $S_p^{\mathbf{a}} F$ for any $\mathbf{a} \in D$. Now for $\mathbf{o} \preceq \mathbf{a}$ and any distinguished open subset $U \subseteq \overline{D}$ we have $U \subseteq (S^{\mathbf{a}})^{+1}(U)$, hence the precosheaf F itself yields a map from $F(U)$ to $S_p^{\mathbf{a}} F(U)$. Now $(S^{\mathbf{a}})^{+1}$ is monotone and thus these maps describe a homomorphism from F to $S_p^{\mathbf{a}} F$.

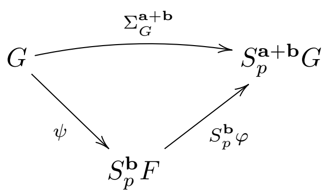
Definition. We denote the previously described homomorphism by $\Sigma_F^{\mathbf{a}} \colon F \to S_p^{\mathbf{a}} F$.

We note that $\Sigma^{\mathbf{a}}$ is a natural transformation from the identity functor id to $S_p^{\mathbf{a}}$. With these definitions in place we can define interleavings of precosheaves. To this end let F and G be precosheaves on $\overline{D}$.

Definition. For $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$ an $(\mathbf{a}, \mathbf{b})$-*interleaving of F and G* is a pair of homomorphisms $\varphi \colon F \to S_p^{\mathbf{a}} G$ and $\psi \colon G \to S_p^{\mathbf{b}} F$ such that



and



commute.

We say *F and G are* $(\mathbf{a}, \mathbf{b})$-*interleaved* if there is an $(\mathbf{a}, \mathbf{b})$-interleaving of F and G.

Now we use the weightings on $\mathcal{D}$ we defined previously to describe two interleaving distances for precosheaves on $\overline{D}$.

Definition. Let $\mathcal{I}$ be the set of all $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$ such that there is an $(\mathbf{a}, \mathbf{b})$-interleaving of F and G.
Then we set $M(F, G) := \inf(\epsilon \circ \gamma \circ \delta)(\mathcal{I})$ and $\mu(F, G) := \inf(\epsilon \circ \delta)(\mathcal{I})$.

We name $M(F, G)$ the *absolute interleaving distance of F and G* and $\mu(F, G)$ the *relative interleaving distance of F and G.*

Now let $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ be continuous functions. At this point the next steps would be to proof the triangle inequalities for M and μ and to show, that the interleaving distances $M(\mathcal{C}f, \mathcal{C}g)$ and $\mu(\mathcal{C}f, \mathcal{C}g)$ really provide lower bounds to $M(f, g)$ respectively $\mu(f, g)$ as we did for join trees. Instead we will derive these results from more general statements however.

Nevertheless we can now repeat another question that we pledged to address in a more precise way. In the beginning we introduced the interleaving distances of join trees $\mathcal{R}\mathcal{E}f$ and $\mathcal{R}\mathcal{E}g$. Above we argued that the functors $\mathcal{R}$ and $\mathcal{C}$ are closely related, so it seems very reasonable to compare the interleavings distances of $\mathcal{R}\mathcal{E}f$ and $\mathcal{R}\mathcal{E}g$ to those of the precosheaves $\mathcal{C}\mathcal{E}f$ and $\mathcal{C}\mathcal{E}g$. Later we will show the following

(9) **Theorem.** If X and Y are smooth and compact manifolds, then

$$M_J(\mathcal{R}\mathcal{E}f, \mathcal{R}\mathcal{E}g) = M(\mathcal{C}\mathcal{E}f, \mathcal{C}\mathcal{E}g)$$

and

$$\mu_J(\mathcal{R}\mathcal{E}f, \mathcal{R}\mathcal{E}g) = \mu(\mathcal{C}\mathcal{E}f, \mathcal{C}\mathcal{E}g).$$

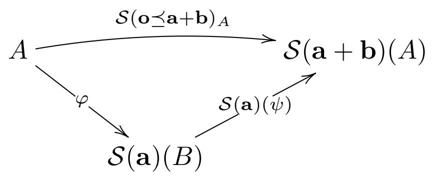
Interleavings in D-Categories

Up to this point we have seen two notions of an interleaving, the first for join trees and the second for precosheaves. In order to show theorem 9 we will use several more and to not repeatedly define new notions with slight modifications we define a common generalization. The idea is to formalize the additional structure, needed on a certain category $\mathbf{C}$, to define the notion of an interleaving between any two objects in $\mathbf{C}$.

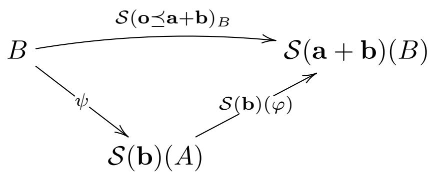
Definition (Strict D-Categories)**.** A *strict D-category* is a category $\mathbf{C}$ with a strict monoidal functor $\mathcal{S}$ from D to the category of endofunctors on $\mathbf{C}$. We refer to $\mathcal{S}$ as the *smoothing functor of* $\mathbf{C}$.

Now let $\mathbf{C}$ be a strict D-category with smoothing functor $\mathcal{S}$ and let A and B be two objects in $\mathbf{C}$. It turns out that the smoothing functor $\mathcal{S}$ is all we need in order to define the notion of an interleaving between A and B.

Definition (Interleavings)**.** For $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$ an $(\mathbf{a}, \mathbf{b})$-*interleaving of A and B* is a pair of homomorphisms $\varphi\colon A \to \mathcal{S}(\mathbf{a})(B)$ and $\psi\colon B \to \mathcal{S}(\mathbf{b})(A)$ such that the diagrams



and



commute.

We say *A and B are $(\mathbf{a}, \mathbf{b})$-interleaved* if there is an $(\mathbf{a}, \mathbf{b})$-interleaving of A and B.

The two interleaving distances of A and B are defined similarly to those of two precosheaves, we spell out the definitions nevertheless.

Definition (Interleaving Distances)**.** Let $\mathcal{I}$ be the set of all $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$ such that there is an $(\mathbf{a}, \mathbf{b})$-interleaving of A and B.
Then we set $M_{\mathcal{S}}(A, B) := \inf(\epsilon \circ \gamma \circ \delta)(\mathcal{I})$ and $\mu_{\mathcal{S}}(A, B) := \inf(\epsilon \circ \delta)(\mathcal{I})$.

In the following example we show that interleavings of precosheaves and the interleaving distances of precosheaves on $\overline{D}$ are an instance of the notions we defined here.

(10) *Example.* Let $\mathcal{B}$ be the intersection-base of $\overline{D}$, then we may identify the monoidal poset of monotone self-maps $\mathrm{End}(\mathcal{B})$ with the category of endofunctors on $\mathcal{B}$. Now let $\mathbf{C}$ be the category of set-valued precosheaves on $\overline{D}$. We define the precomposition functor

$$\tilde{\mathcal{S}}\colon \mathrm{End}(\mathcal{B}) \to \mathrm{End}(\mathbf{C}), \begin{cases} T \mapsto (F \mapsto F \circ T) \\ \eta \mapsto (F \mapsto F \circ \eta). \end{cases}$$

We note that $\tilde{\mathcal{S}}$ is a strict monoidal functor. Moreover the map

$$\left(S^{(\_)}\right)^{+1}\colon D \to \mathrm{End}(\mathcal{B}), \mathbf{a} \mapsto (S^{\mathbf{a}})^{+1}$$

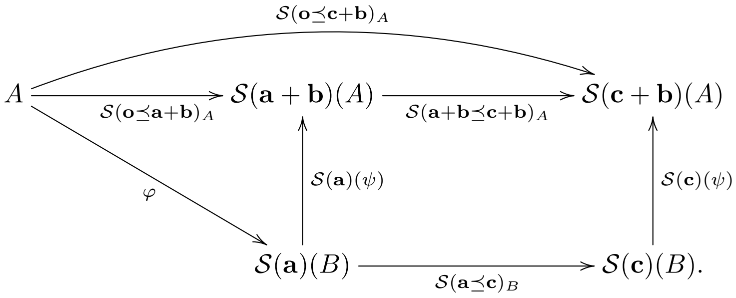
is a homomorphism of monoidal posets, hence the functor $\mathcal{S} := \tilde{\mathcal{S}} \circ \left(S^{(\_)}\right)^{+1}$ is strict monoidal as well. Now we observe that $\mathcal{S}(\mathbf{a}) = S^{\mathbf{a}}_p$ for $\mathbf{a} \in D$. And if $\mathbf{o} \preceq \mathbf{a}$, then $\mathcal{S}(\mathbf{o} \preceq \mathbf{a}) = \Sigma^{\mathbf{a}}$.

Properties of Interleavings

Now let $\mathbf{C}$ be a strict D-category with smoothing functor $\mathcal{S}$ and let A and B be objects of $\mathbf{C}$. We aim to provide some useful properties about interleavings in $\mathbf{C}$.

Lemma (Monotonicity)**.** For $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$ let $\varphi\colon A \to \mathcal{S}(\mathbf{a})(B)$ and $\psi\colon B \to \mathcal{S}(\mathbf{b})(A)$ be an $(\mathbf{a}, \mathbf{b})$-interleaving of A and B. Now suppose we have $\mathbf{c} \in D$ with $\mathbf{a} \preceq \mathbf{c}$. Then $\mathcal{S}(\mathbf{a} \preceq \mathbf{c})_B \circ \varphi$ and ψ form an $(\mathbf{c}, \mathbf{b})$-interleaving of A and B. Completely analogously $\mathcal{S}(\mathbf{b} \preceq \mathbf{d})_A \circ \psi$ and φ yield an $(\mathbf{a}, \mathbf{d})$-interleaving for any $\mathbf{d} \in D$ with $\mathbf{b} \preceq \mathbf{d}$.

Proof. We consider the diagram



The upper triangle commutes because $\mathcal{S}$ is a functor and the triangle on the left commutes by assumption. Further we have $\mathcal{S}(\mathbf{a}+\mathbf{b}\preceq\mathbf{c}+\mathbf{b}) = \mathcal{S}(\mathbf{a}\preceq\mathbf{c})\circ\mathcal{S}(\mathbf{b})$, since $\mathcal{S}$ is strict monoidal and thus the square commutes as $\mathcal{S}(\mathbf{a}\preceq\mathbf{c})$ is a natural transformation. Since all the inner triangles and the square commute in the above diagram, the outer triangle commutes as well. The other triangle, for $\mathcal{S}(\mathbf{a}\preceq\mathbf{c})_B\circ\varphi$ and ψ to be an interleaving, also commutes, because we have $\mathcal{S}(\mathbf{a}+\mathbf{b}\preceq\mathbf{c}+\mathbf{b})_B\circ\mathcal{S}(\mathbf{o}\preceq\mathbf{a}+\mathbf{b})_B = \mathcal{S}(\mathbf{o}\preceq\mathbf{c}+\mathbf{b})_B$.

(11) **Corollary.** Let $\mathcal{I}$ be the set of all $(\mathbf{a},\mathbf{b})\in\mathcal{D}$ such that A and B are $(\mathbf{a},\mathbf{b})$-interleaved. Further let $\mathcal{I}' := \mathcal{I}\cap\triangledown$ and $\mathcal{I}'' := \mathcal{I}\cap\blacktriangledown$. Then we have $\mu_{\mathcal{S}}(A,B) = \inf\epsilon(\mathcal{I}')$ and $M_{\mathcal{S}}(A,B) = \inf(\epsilon\circ\gamma)(\mathcal{I}') = \inf\epsilon(\mathcal{I}'')$.

Proof. First we note that $\mathcal{I}'\subseteq\delta(\mathcal{I})$, since $\delta|_{\triangledown} = \mathrm{id}_{\triangledown}$. By the previous lemma we also have $\mathcal{I}'\supseteq\delta(\mathcal{I})$, hence $\mathcal{I}' = \delta(\mathcal{I})$. This implies $\mu_{\mathcal{S}}(A,B) = \inf\epsilon(\mathcal{I}')$ and $M_{\mathcal{S}}(A,B) = \inf(\epsilon\circ\gamma)(\mathcal{I}')$. Similarly we have $\mathcal{I}'' = \gamma(\mathcal{I}')$ and thus $M_{\mathcal{S}}(A,B) = \inf\epsilon(\mathcal{I}'')$.

We will now use this corollary to give more concise descriptions of the interleaving distances. The reason we didn't define the interleaving distances with these more concise descriptions in the first place is that we believe, having those additional interleavings around, can help with the computation of the interleaving distances. Now suppose $(a,b;c,d)\in\triangledown$, then $(a,b;c,d) = (-d,b;-b,d)$. Thus we have the bijection

$$\Phi\colon \triangledown\to D^{\perp}, (a,b;c,d)\mapsto(d,b)$$

with inverse

$$\Psi\colon D^{\perp}\to\triangledown, (a,b)\mapsto(-a,b;-b,a).$$

Definition. For $(a,b)\in D^{\perp}$ an *(a,b)-interleaving of A and B* is a $(-a,b;-b,a)$-interleaving of A and B.

If there is an (a,b)-interleaving of A and B, we say *A and B are (a,b)-interleaved.*

With this definition we get a corollary to the previous corollary.

(12) **Corollary.** Let $\mathcal{J}$ be the set of all $(a,b)\in D^{\perp}$ such that A and B are (a,b)-interleaved, then $\mu_{\mathcal{S}}(A,B) = \inf\epsilon'(\mathcal{J})$ and $M_{\mathcal{S}}(A,B) = \inf\epsilon''(\mathcal{J})$.

Proof. Let $\mathcal{I}'$ be as in the previous corollary. By the above observations we have $\mathcal{I}' = \Psi(J)$, thus in conjunction with the previous corollary $\mu_{\mathcal{S}}(A, B) = \inf(\epsilon \circ \Psi)(\mathcal{J})$ and $M_{\mathcal{S}}(A, B) = \inf(\epsilon \circ \gamma \circ \Psi)(\mathcal{J})$. Now let $(a, b) \in D^{\perp}$ and $\varepsilon = \max\{a, b\}$, then we have $(\epsilon \circ \Psi)((a, b)) = \epsilon(-a, b; -b, a) = \frac{1}{2}(b + a) = \epsilon'(a, b)$ and

$$\begin{aligned}(\epsilon \circ \gamma \circ \Psi)(a, b) &= (\epsilon \circ \gamma)(-a, b; -b, a) \\ &= \epsilon(-\varepsilon, \varepsilon; -\varepsilon, \varepsilon) \\ &= \varepsilon = \max\{a, b\} = \epsilon''(a, b).\end{aligned}$$

We further simplify the absolute interleaving distance of A and B.

Defintion (ε-Interleaving)**.** For $\varepsilon \geq 0$ an *ε-interleaving of A and B* is an $(\varepsilon, \varepsilon)$-interleaving of A and B.

If there is an ε-interleaving of A and B, we say *A and B are ε-interleaved.*

Now an $(\varepsilon, \varepsilon)$-interleaving of A and B is just a $(-\varepsilon, \varepsilon; -\varepsilon, \varepsilon)$-interleaving. Further we have $\epsilon((-\varepsilon, \varepsilon; -\varepsilon, \varepsilon)) = \varepsilon$. And as noted earlier $\epsilon|_{\blacktriangledown}$ is a bijection. Thus we have yet another corollary to corollary 11.

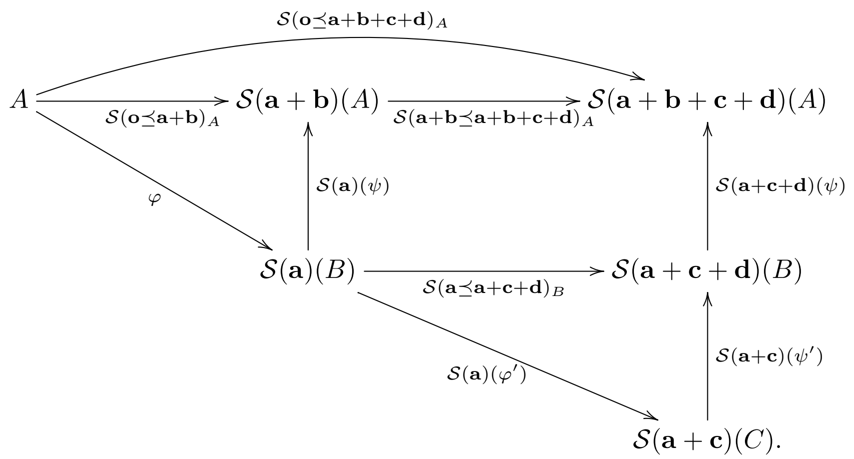
(13) **Corollary.** Let $\mathcal{V}$ be the set of all $\varepsilon \geq 0$ such that A and B are ε-interleaved. Then $M_{\mathcal{S}}(A, B) = \inf \mathcal{V}$.

Proof. Let $\mathcal{I}''$ be as in the corollary 11. By the above observations we have $\mathcal{I}'' = (\epsilon|_{\blacktriangledown})^{-1}(\mathcal{V})$. In conjunction with the corollary 11 we get $M_{\mathcal{S}}(A, B) = \inf \epsilon\big((\epsilon|_{\blacktriangledown})^{-1}(\mathcal{V})\big) = \inf \mathcal{V}$.

Next we proof a type of triangle inequality for both interleaving distances. To this end let C be another object of $\mathbf{C}$.

Lemma. Let $(\mathbf{a}, \mathbf{b}), (\mathbf{c}, \mathbf{d}) \in \mathcal{D}$. Further let $\varphi\colon A \to \mathcal{S}(\mathbf{a})(B)$ and $\psi\colon B \to \mathcal{S}(\mathbf{b})(A)$ be an $(\mathbf{a}, \mathbf{b})$-interleaving of A and B and let $\varphi'\colon B \to \mathcal{S}(\mathbf{c})(C)$ and $\psi'\colon C \to \mathcal{S}(\mathbf{d})(B)$ be a $(\mathbf{c}, \mathbf{d})$-interleaving of B and C. Then $\mathcal{S}(\mathbf{a})(\varphi') \circ \varphi$ and $\mathcal{S}(\mathbf{d})(\psi) \circ \psi'$ form an $(\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{d})$-interleaving of A and C.

Proof. We consider the diagram



Now the upper triangle commute because $\mathcal{S}$ is a functor and the triangle on the left commutes by assumption. Further we have $\mathcal{S}(\mathbf{a}) \circ \mathcal{S}(\mathbf{o} \preceq \mathbf{c}+\mathbf{d}) = \mathcal{S}(\mathbf{a} \preceq \mathbf{a}+\mathbf{c}+\mathbf{d})$, since $\mathcal{S}$ is strict monoidal and thus the lower triangle commutes. For the square to commute, we observe that $\mathcal{S}(\mathbf{a}+\mathbf{b} \preceq \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}) = \mathcal{S}(\mathbf{a} \preceq \mathbf{a}+\mathbf{c}+\mathbf{d}) \circ \mathcal{S}(\mathbf{b})$, again because $\mathcal{S}$ is strict monoidal. Since all inner triangles and the square commute in the above diagram, the outer triangle commutes as well. Now $\mathcal{S}(\mathbf{a}+\mathbf{c}+\mathbf{d}) = \mathcal{S}(\mathbf{a}+\mathbf{c}) \circ \mathcal{S}(\mathbf{d})$ once again because $\mathcal{S}$ is strict monoidal, hence

$$\begin{aligned}\mathcal{S}(\mathbf{a}+\mathbf{c})(\mathcal{S}(\mathbf{d})(\psi) \circ \psi') &= \mathcal{S}(\mathbf{a}+\mathbf{c})(\mathcal{S}(\mathbf{d})(\psi)) \circ \mathcal{S}(\mathbf{a}+\mathbf{c})(\mathcal{S}(\psi') \\ &= \mathcal{S}(\mathbf{a}+\mathbf{c}+\mathbf{d})(\psi) \circ \mathcal{S}(\mathbf{a}+\mathbf{c})(\mathcal{S}(\psi').\end{aligned}$$

If we now consider the large triangle on it's own and we substitute the two vertical maps on the right with the left hand side of the previous equation, then we obtain the first of the two triangles for $\mathcal{S}(\mathbf{a})(\varphi') \circ \varphi$ and $\mathcal{S}(\mathbf{d})(\psi) \circ \psi'$ to be an interleaving. The proof that the second triangle commutes is completely analogous.

(14) **Corollary** (Triangle Inequality)**.** We have

$$M_{\mathcal{S}}(A, C) \leq M_{\mathcal{S}}(A, B) + M_{\mathcal{S}}(B, C)$$

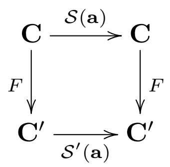
and

$$\mu_{\mathcal{S}}(A, C) \leq \mu_{\mathcal{S}}(A, B) + \mu_{\mathcal{S}}(B, C).$$

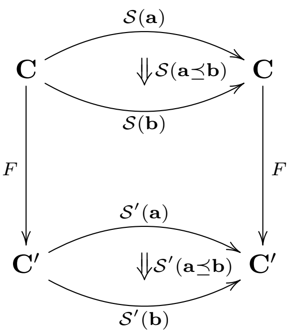
Homomorphisms of *D*-Categories

After introducing the notion of a strict D-category, we seek ways to relate different D-categories and their interleavings to each other. To this end let $\mathbf{C}$ and $\mathbf{C}'$ be strict D-categories with smoothing functor $\mathcal{S}$ respectively $\mathcal{S}'$.

Definition (1-Homomorphism)**.** A 1-*homomorphism of strict D-categories from* $\mathbf{C}$ *to* $\mathbf{C}'$ is a functor $F \colon \mathbf{C} \to \mathbf{C}'$ such that



and



commute for all $\mathbf{a}, \mathbf{b} \in D$ with $\mathbf{a} \preceq \mathbf{b}$.

Remark. We read the second diagram of the above definition as $F \circ \mathcal{S}(\mathbf{a} \preceq \mathbf{b}) = \mathcal{S}'(\mathbf{a} \preceq \mathbf{b}) \circ F$.

Now let $F \colon \mathbf{C} \to \mathbf{C}'$ be a 1-homomorphism and let A and B be objects of $\mathbf{C}$.

Lemma. Let $\varphi \colon A \to \mathcal{S}(\mathbf{a})(B)$ and $\psi \colon B \to \mathcal{S}(\mathbf{b})(A)$ be an $(\mathbf{a}, \mathbf{b})$-interleaving for some $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$, then $F(\varphi)$ and $F(\psi)$ form an $(\mathbf{a}, \mathbf{b})$-interleaving in $\mathbf{C}'$.

(15) **Corollary.** We have $M_{\mathcal{S}'}(F(A), F(B)) \leq M_{\mathcal{S}}(A, B)$ and $\mu_{\mathcal{S}'}(F(A), F(B)) \leq \mu_{\mathcal{S}}(A, B)$.

Lemma. Now suppose that F is faithful as a functor from $\mathbf{C}$ to $\mathbf{C}'$ and that $\varphi \colon A \to \mathcal{S}(\mathbf{a})(B)$ and $\psi \colon B \to \mathcal{S}(\mathbf{b})(A)$ are arbitrary homomorphisms for some $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$, with $F(\varphi)$ and $F(\psi)$ an $(\mathbf{a}, \mathbf{b})$-interleaving of $F(A)$ and $F(B)$. Then φ and ψ are an $(\mathbf{a}, \mathbf{b})$-interleaving as well.

The previous two lemmata have the following

Corollary. If F is full and faithful, then F induces a bijection between the $(\mathbf{a}, \mathbf{b})$-interleavings of A and B and the $(\mathbf{a}, \mathbf{b})$-interleavings of $F(A)$ and $F(B)$ for any $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$.

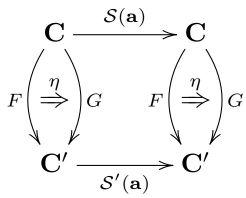
Corollary. If F is full and faithful we have $M_{\mathcal{S}'}(F(A), F(B)) = M_{\mathcal{S}}(A, B)$ and $\mu_{\mathcal{S}'}(F(A), F(B)) = \mu_{\mathcal{S}}(A, B)$.

We also note that we can compose 1-homomorphisms. To this end let $\mathbf{C}''$ be another strict D-category.

Lemma. Let $G \colon \mathbf{C}' \to \mathbf{C}''$ be another 1-homomorphism, then $G \circ F$ is a 1-homomorphism from $\mathbf{C}$ to $\mathbf{C}''$.

Now we defined 1-homomorphisms as functors with special properties. For any two functors F and G from $\mathbf{C}$ to $\mathbf{C}'$ we have the class of natural transformations from F to G. Now suppose F and G are 1-homomorphisms, then F and G are in some sense compatible with $\mathcal{S}$ and $\mathcal{S}'$. We name a natural transformation from F to G, that is compatible with $\mathcal{S}$ and $\mathcal{S}'$, a 2-*homomorphism from F to G*.

Definition (2-Homomorphism) **.** A 2-*homomorphism of strict D-categories from F to G* is a natural transformation $\eta \colon F \Rightarrow G$ such that



commutes for all $\mathbf{a} \in D$.

Remark. We read the diagram of the previous definition as $\eta \circ \mathcal{S}(\mathbf{a}) = \mathcal{S}'(\mathbf{a}) \circ \eta$. This definition does not include the condition $\eta \circ \mathcal{S}(\mathbf{a} \preceq \mathbf{b}) = \mathcal{S}'(\mathbf{a} \preceq \mathbf{b}) \circ \eta$ for $\mathbf{a} \preceq \mathbf{b}$, where $\circ$ is the Godement product of natural transformations. The reason is, that this equation follows from the previous two definitions. If we choose one of the two formulas for the Godement product of η and $\mathcal{S}(\mathbf{a} \preceq \mathbf{b})$ and rewrite the term using the previous two definitions, then we obtain the other of the two formulas for the Godement product of $\mathcal{S}'(\mathbf{a} \preceq \mathbf{b})$ and η.

Just like we can compose natural transformations we can also compose 2-homomorphisms.

Lemma. Let $\eta \colon F \Rightarrow G$ and $\theta \colon G \Rightarrow H$ be 2-homomorphisms, then $\theta \circ \eta$ is a 2-homomorphism as well.

Moreover we can compose 1-homomorphisms with 2-homomorphisms and vice versa.

Lemma. Let F and G be 1-homomorphisms from $\mathbf{C}$ to $\mathbf{C}'$, let $\eta \colon F \Rightarrow G$ be a 2-homomorphism, and let H be a 1-homomorphism from $\mathbf{C}'$ to $\mathbf{C}''$, then the composition $H \circ \eta$, pictorially

$$\mathbf{C} \underset{G}{\overset{F}{\rightrightarrows}}\!\!\Downarrow\eta\; \mathbf{C}' \xrightarrow{H} \mathbf{C}'',$$

is 2-homomorphism from $H \circ F$ to $H \circ G$.

Similarly if F and G are 1-homomorphisms from $\mathbf{C}'$ to $\mathbf{C}''$ and if H is a 1-homomorphism from $\mathbf{C}$ to $\mathbf{C}'$, then the composition $\eta \circ H$, pictorially

$$\mathbf{C} \xrightarrow{H} \mathbf{C}' \underset{G}{\overset{F}{\rightrightarrows}}\!\!\Downarrow\eta\; \mathbf{C}'',$$

is again a 2-homomorphism.

Remark. Altogether we obtain the structure of a strict 2-category on the collection of all strict D-categories.

Positive Persistence-Enhancements

In the previous section we defined strict D-categories, interleavings of objects in D-categories, and showed that interleavings of precosheaves are an instance of this. Then we derived some properties about the interleaving distances, such as the triangle inequality and this compensates for the triangle inequality having been left out in the section before. However we did not prove, that the interleaving distances of Reeb precosheaves provide lower bounds to the corresponding distances of spaces. And it is the aim of this section to make up for this deficit. In the previous section we already observerd that 1-homomorphisms of D-categories yield inequalities for the corresponding interleaving distances. To use this result we will take two steps. First we will embed the category of $\mathbb{R}$-spaces into a D-category $\mathbf{F}$, so that the interleaving distances on the image of this embedding are equal to the corresponding distances on $\mathbb{R}$-spaces. Second we factor $\mathcal{C}$ into this embedding and a 1-homomorphism $\tilde{\mathcal{C}}$ from $\mathbf{F}$ to the category of precosheaves on $\overline{D}$.

Definition. Here we define the category $\mathbf{F}$.

The class of objects of $\mathbf{F}$ is the class of all pairs $(f, \mathbf{a})$, where $f\colon X \to \mathbb{R}$ is a continuous real-valued function and $\mathbf{a} \in D$.

Next we specify the homomorphsims of $\mathbf{F}$. To this end let $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ be continuous functions and $(a, b), (c, d) \in D$. Then a *homomorphism from* $(f, (a, b))$ *to* $(g, (c, d))$ is a continuous map $\varphi\colon X \to Y$ such that $b - d \leq f(p) - g(\varphi(p)) \leq a - c$ for all $p \in X$.

For $b = d = \infty$ we read the left hand side of the previous inequality as $-\infty$ and for $a = c = -\infty$ we read the right hand side to be ∞.

So far we have the category $\mathbf{F}$ and the full embedding

$$(\_, \mathbf{o})\colon \begin{cases} f \mapsto (f, \mathbf{o}) \\ \varphi \mapsto \varphi. \end{cases}$$

Next we provide a smoothing functor for $\mathbf{F}$.

Definition. Let $f\colon X \to \mathbb{R}$ be a continuous function and let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in D$ with $\mathbf{b} \preceq \mathbf{c}$. Then we set $\mathcal{T}(\mathbf{b})((f, \mathbf{a})) := (f, \mathbf{a} + \mathbf{b})$ and $\mathcal{T}(\mathbf{b} \preceq \mathbf{c})_{(f,\mathbf{a})} := \mathrm{id}_X$.

The previous definition augments $\mathbf{F}$ with the structure of a strict D-category. Now we compare the distances of functions to the interleaving distances of $\mathbf{F}$. To this end let $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ be continuous functions.

Lemma. Let $r \in \mathbb{R}$, $\varepsilon \geq 0$, and let $\varphi\colon X \to Y$ be a homeomorphism. Then we have $-\varepsilon \leq r + f(p) - g(\varphi(p)) \leq \varepsilon$ if and only if φ and φ^{-1} form an $(-r+\varepsilon, r+\varepsilon)$-interleaving of $(f, \mathbf{o})$ and $(g, \mathbf{o})$.

Corollary. We have $M(f, g) = M_{\mathcal{T}}((f, \mathbf{o}), (g, \mathbf{o}))$ and $\mu(f, g) = \mu_{\mathcal{T}}((f, \mathbf{o}), (g, \mathbf{o}))$.

Proof. The first equation follows in conjunction with corollary 13, see the section on properties of interleavings, and the second equation follows in conjunction with corollary 12.

Now we have an embedding of the category of $\mathbb{R}$-spaces into $\mathbf{F}$, that preserves the distances. In example 10 from the previous section we augmented the category of set-valued precosheaves on $\overline{D}$ with the structure of a D-category and we observed that the interleavings of precosheaves are precisely the interleavings with respect to this structure. Our next and final step towards a lower bound is to provide a 1-homomorphism $\tilde{\mathcal{C}}$ from $\mathbf{F}$ to the category of set-valued precosheaves on $\overline{D}$ such that $\mathcal{C} = \tilde{\mathcal{C}}((\_, \mathbf{o}))$. Applying the same procedure to an arbitrary functor F on the category of $\mathbb{R}$-spaces is what we name a *positive persistence-enhancement for* F. Now let F be a functor from the category of $\mathbb{R}$-spaces to some category $\mathbf{C}$

Definition (Positive Persistence-Enhancement)**.** A *postive persistence-enhancement for* F is the structure of a strict D-category on $\mathbf{C}$ together with a 1-homomorphism $\tilde{F}$ from $\mathbf{F}$ to $\mathbf{C}$ such that $F = \tilde{F}((\_, \mathbf{o}))$.

(16) **Propostion.** If there is a positive persistence-enhancement of F with smoothing functor $\mathcal{S}$ on $\mathbf{C}$, then we have $M_{\mathcal{S}}(F(f), F(g)) \leq M(f, g)$ and $\mu_{\mathcal{S}}(F(f), F(g)) \leq \mu(f, g)$.

Proof. This follows from corollary 15, see the section on homomorphisms in D-categories, and the previous corollary.

Example. As a first example we provide a positive persistence-enhancements for the Reeb precosheaf $\mathcal{C}$. In the previous section we already defined the structure of a strict D-category on the category of precosheaves. So the only thing left to define for a positive persistence-enhancement is the functor $\tilde{\mathcal{C}}$. To this end we set $\Delta^{\mathbf{a}} \colon \mathbb{R} \to \overline{\mathbb{E}}, t \mapsto (t, t) - \mathbf{a}$ and $\overline{\mathcal{C}}((f, \mathbf{a})) := (\Delta^{\mathbf{a}} \circ f)_* \Lambda$ for any $\mathbf{a} \in D$. Now let $\varphi \colon (f, \mathbf{a}) \to (g, \mathbf{b})$ be a homomorphism in $\mathbf{F}$ for some $\mathbf{a}, \mathbf{b} \in D$ and let $U \subseteq \overline{\mathbb{E}}$ be a distinguished open subset, then we have $\varphi((\Delta^{\mathbf{a}} \circ f)^{-1}(U)) \subseteq (\Delta^{\mathbf{b}} \circ g)^{-1}(U)$ and thus the restriction $\varphi|_{(\Delta^{\mathbf{a}} \circ f)^{-1}(U)} \colon (\Delta^{\mathbf{a}} \circ f)^{-1}(U) \to (\Delta^{\mathbf{b}} \circ g)^{-1}(U)$. We set $\overline{\mathcal{C}}(\varphi)_U := \Lambda\big(\varphi|_{(\Delta^{\mathbf{a}} \circ f)^{-1}(U)}\big)$. This defines the functor $\overline{\mathcal{C}}$ from $\mathbf{F}$ to the category of set-valued precosheaves on $\overline{\mathbb{E}}$. Now let $i \colon \overline{D} \subseteq \overline{\mathbb{E}}$ denote the inclusion, then the composition $\tilde{\mathcal{C}} := i_p \circ \overline{\mathcal{C}}$ defines a 1-homomorphism of strict D-categories with $\mathcal{C} = \tilde{\mathcal{C}}((\_, \mathbf{o}))$.

Now let F and G be functors from the category of $\mathbb{R}$-spaces to some D-category $\mathbf{C}$ with smoothing functor $\mathcal{S}$ and let $\eta \colon F \to G$ be a natural transformation. We consider η a functor from the category of $\mathbb{R}$-spaces to the category of arrows in $\mathbf{C}$. Moreover we consider the category of arrows in $\mathbf{C}$ a D-category with smoothing functor $\mathcal{S}$. (We just apply the smoothing functor to the homomorphisms.) Then a (positive) persistence-enhancement of η with smoothing functor $\mathcal{S}$ is already determined by the corresponding enhancements for F and G. Now suppose $\tilde{F}$ and $\tilde{G}$ are arbitrary persistence-enhancements for F and G both with smoothing functor $\mathcal{S}$.

Definition. We say $\mathcal{S}$*,* $\tilde{F}$*, and* $\tilde{G}$ *combine to a persistence-enhancement of* η if the map $(f, \mathbf{a}) \mapsto (\mathcal{S}(\mathbf{a}) \circ \eta)_f$ is a natural transformation from $\tilde{F}$ to $\tilde{G}$.

(17) *Remark.* If $\mathcal{S}$, $\tilde{F}$, and $\tilde{G}$ combine to a persistence-enhancement of η, then $(f, \mathbf{a}) \mapsto (\mathcal{S}(\mathbf{a}) \circ \eta)_f$ is a 2-homomorphism from $\tilde{F}$ to $\tilde{G}$.

Complete Persistence-Enhancements

In the previous section we defined positive persistence enhancements of functors on $\mathbb{R}$-spaces and provided one for $\mathcal{C}$, thereby finally establishing that the interleaving distances of the Reeb precosheaves provide lower bounds to the correspoding distances on functions. The next aim is to connect this to the interleaving distances of join-trees and to proof theorem 9. Unfortunately there is a bit of a problem with our use of infinity. If we consider the diagrams for interleavings in D-categories, then there is a smoothing functor on the top right but none on the top left. In the diagrams for interleavings of join trees however there is a shift on the top left but none on the top right. If it wasn't possible for a or b to attain the value ∞, then it wouldn't matter if we shift on the left or on the right. But if a or b is infinity, then the corresponding shift on the right is ill-defined. The purpose of including infinity is that we get an ∞-interleaving from any other interleaving by monotonicity. So to determine the interleaving distances we could start by considering all ∞-interleavings and then optimize them with respect to the two weightings. The downside is that in order to harness this framework for comparing the interleaving distance of the Reeb precosheaf to that of the join tree, we have to introduce some more terminology.

Above we defined the monoidal poset D. Now the partial order $\preceq$ canonically extends to $\overline{\mathbb{E}}$ but for the monoidal operation there is some ambiguity when adding ∞ and $-\infty$. But it is still possible if we give up commutativity. More specifically we specify $-\infty + \infty := \infty$ and $\infty - \infty := -\infty$. So the last term always dominates. Now $\overline{\mathbb{E}}$ contains D as a submonoid. Moreover $-D$ is a commutative submonoid of $\overline{\mathbb{E}}$ as well and negation yields an order-reversing monoid isomorphism from D to $-D$. Similarly to D-categories we now define $-D$- and $\overline{\mathbb{E}}$-categories.

Definition. A *strict* $\overline{\mathbb{E}}$*-category* respectively $-D$*-category* is a category $\mathbf{C}$ with a strict monoidal functor $\mathcal{S}$ from $\overline{\mathbb{E}}$ respectively $-D$ to the category of endofunctors on $\mathbf{C}$. We refer to $\mathcal{S}$ as the *smoothing functor of* $\mathbf{C}$.

Remark. If $\mathbf{C}$ is a strict $-D$-category with smoothing functor $\mathcal{S}$, then the opposite category $\mathbf{C}^{\mathrm{op}}$ is a strict D-category with smoothing functor $\mathcal{S}(-(\_))$.

Now we define interleavings in $-D$-categories. To this end let $\mathbf{C}$ be a $-D$-category with smoothing functor $\mathcal{S}$ and let A and B be objects of $\mathbf{C}$.

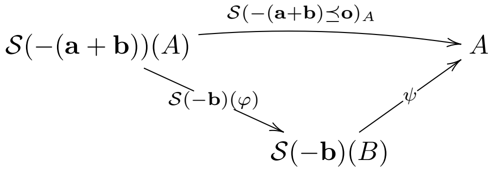
Definition. For $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$ an $(\mathbf{a}, \mathbf{b})$*-interleaving of* A *and* B *in* $\mathbf{C}$ is an $(\mathbf{a}, \mathbf{b})$-interleaving of B and A in $\mathbf{C}^{\mathrm{op}}$ with respect to the smoothing functor $\mathcal{S}(-(\_))$.

We say A *and* B *are* $(\mathbf{a}, \mathbf{b})$*-interleaved* if there is an $(\mathbf{a}, \mathbf{b})$-interleaving of A and B.

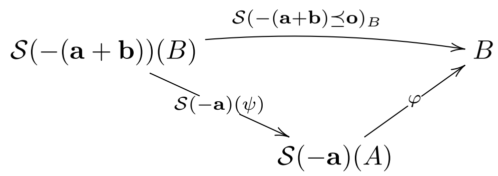
Similarly an (a, b)*-interleaving of* A *and* B *in* $\mathbf{C}$ is an (a, b)-interleaving of B and A in $\mathbf{C}^{\mathrm{op}}$ for $(a, b) \in D^{\perp}$.

For convenience we spell out the meaning of the previous definition.

Remark. For $(\mathbf{a}, \mathbf{b}) \in \mathcal{D}$ an $(\mathbf{a}, \mathbf{b})$-interleaving of A and B in $\mathbf{C}$ is a pair of homomorphisms $\varphi \colon \mathcal{S}(-\mathbf{a})(A) \to B$ and $\psi \colon \mathcal{S}(-\mathbf{b})(B) \to A$ such that the diagrams



and



commute.

With these definitions we may talk about the absolute and relative interleaving distance of A and B. Now let us assume $\mathbf{C}$ is a strict $\overline{\mathbb{E}}$-category with smoothing functor $\mathcal{S}$, then it also is a D- and a $-D$-category. So if we don't mention, whether we work with $\mathbf{C}$ as a D-category or as a $-D$-category, then our term *interleaving* is ambiguous. Here (and with the next lemma) we argue that this creates no problem. Suppose we have $\varphi\colon A \to \mathcal{S}(\mathbf{a})(B)$ for some $\mathbf{a} \in D$, then we get the homomorphism $\varphi^b := \mathcal{S}(\mathbf{a} - \mathbf{a} \preceq \mathbf{o})_B \circ \mathbf{S}(-\mathbf{a})(\varphi)$ from $\mathcal{S}(-\mathbf{a})(A)$ to B. And if we now apply the functor $\mathcal{S}(\mathbf{a})$ to φ^b and precompose with $\mathcal{S}(\mathbf{o} \preceq -\mathbf{a} + \mathbf{a})_A$, then we reobtain φ, using that $\mathcal{S}$ is monoidal. Such considerations lead us to the following

(18) **Lemma.** The interleavings of A and B with respect to the D-category sructure of $\mathbf{C}$ are in a canonical bijection with the interleavings of A and B with respect to the $-D$-category sructure.

Homomorphisms of $-D$- and $\overline{\mathbb{E}}$-categories are defined completely analogously to those of D-categories.

Definition. Here we define the categories $-\mathbf{F}$ and $\pm\mathbf{F}$.

The class of objects of $-\mathbf{F}$ respectively $\pm\mathbf{F}$ is the class of all pairs $(f, \mathbf{a})$, where $f\colon X \to \mathbb{R}$ is a continuous real-valued function and $\mathbf{a} \in -D$ respectively $\mathbf{a} \in \overline{\mathbb{E}}$.

Next we specify the homomorphsims of $\mathbf{F}$. To this end let $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ be continuous functions and $(a, b), (c, d) \in \overline{\mathbb{E}}$. Then a *homomorphism from* $(f, (a, b))$ *to* $(g, (c, d))$ is a continuous map $\varphi\colon X \to Y$ such that $b - d \leq f(p) - g(\varphi(p)) \leq a - c$ for all $p \in X$. Whenever there is any ambiguity interpreting the left-hand side of this inequality, we interpret it as $-\infty$, for the right-hand side as ∞.

Next we define negative and complete persistence-enhancements. To this end let F be a functor from the category of $\mathbb{R}$-spaces to some category $\mathbf{C}$.

Definition. A *negative* respectively a *complete persistence-enhancement of* F is the structure of a strict $-D$-category respectively $\overline{\mathbb{E}}$-category on $\mathbf{C}$ together with a 1-homomorphism $\tilde{F}$ from $-\mathbf{F}$ resepctively $\pm\mathbf{F}$ to $\mathbf{C}$ such that $F = \tilde{F}((\_, \mathbf{o}))$.

Some Equivalences

In this section we move one step closer to proving theorem 9. We consider interleavings of precosheaves in the image of the functor $\mathcal{CE}$ and transform those into interleavings somewhat closer to those of join trees.

Equivalence to descending Precosheaf

We start by describing the structure of an $\overline{\mathbb{E}}$-category on the the category of set-valued precosheaves on $\overline{\mathbb{R}}_{-\infty}$. We proceed similar to example 10. To this end let $\mathcal{U}$ be the topology or intersection-base of $\overline{\mathbb{R}}_{-\infty}$ (here they are the same), $\mathcal{Q}$ the intersection-base of $\overline{D}$, and let

$$\overline{s}\colon \overline{\mathbb{E}} \mapsto \mathrm{End}(\mathcal{U}), (a,b) \mapsto \begin{cases} (s^b)^{+1} & b > -\infty \\ (s^{-b})^{-1} & b < \infty. \end{cases}$$

We note that this definition is not over-determined and that $\overline{s}$ is monotone. Now post-composition with the precomposition functor $\tilde{\mathcal{S}}$ yields the smoothing functor $\overline{\mathcal{S}} := \tilde{\mathcal{S}} \circ \overline{s}$. We note that $\overline{\mathcal{S}}((a,b)) = s_p^b$ for $b > -\infty$ and $\overline{\mathcal{S}}((a,b)) = s_*^{-b}$ for $b < \infty$. Now we have the continuous map π^2 from $\overline{D}$ to $\overline{\mathbb{R}}_{-\infty}$ and we aim to show that π_*^2 and π_p^2 are 1-homomorphisms between the corresponding D-categories and that η^{π^2} is a 2-homomorphism. For $(a,b) \in D$ we have $\pi^2 \circ S^{(a,b)} = s^b \circ \pi^2$ and with this we convince ourselves that the diagram

$$\begin{array}{ccccc}
\mathcal{Q} & \xrightarrow{(\pi^2)^{+1}} & \mathcal{U} & \xrightarrow{(\pi^2)^{-1}} & \mathcal{Q} \\
\downarrow{\scriptstyle \left(S^{(a,b)}\right)^{+1}} & & \downarrow{\scriptstyle (s^b)^{+1}} & & \downarrow{\scriptstyle \left(S^{(a,b)}\right)^{+1}} \\
\mathcal{Q} & \xrightarrow[(\pi^2)^{+1}]{} & \mathcal{U} & \xrightarrow[(\pi^2)^{-1}]{} & \mathcal{Q}
\end{array}$$

commutes for all $(a,b) \in D$. (We note that lemma 5 is the commutativity of the left square.)

Lemma. The functor π_*^2 is a 1-homomorphism of D-categories.

Proof. This follows from the commutativity of the right square in the above diagram and the monotonicity of post-composition by $(\pi^2)^{-1}$.

Lemma. The functor π_p^2 is a 1-homomorphism of D-categories.

Proof. This follows from the commutativity of the left square in the above diagram and the monotonicity of post-composition by $(\pi^2)^{+1}$.

Lemma. The natural transformation η^{π^2} is a 2-homomorphism of D-categories.

Proof. This follows from the commutativity of the outer square in the above diagram.

The previous three lemmata have the following

Corollary. The full subcategory of set-valued precosheaves F on $\overline{D}$, with $\eta_F^{\pi^2}$ an isomorphism, is a sub-D-category in the sense that it is invariant under $\mathcal{S}(\mathbf{a})$ for all $\mathbf{a} \in D$.

Now the functor π_*^2 is full and faithful on the category of precosheaves F, with $\eta_F^{\pi^2}$ an isomorphism. Since this is also a D-category and π_*^2 is a 1-homomorphism on this D-category, π_*^2 yields bijections of interleavings.

Now let $f\colon X \to \mathbb{R}$ a continuous function.

(19) **Lemma.** The homomorphism $(\eta^{\pi^2} \circ \mathcal{C} \circ \mathcal{E})_f$ from $\mathcal{C}\mathcal{E}f$ to $\pi_p^2 \pi_*^2 \mathcal{C}\mathcal{E}f$ is an isomorphism.

Proof. For $a < r < b$ we set $U := (a, \infty] \times [-\infty, b)$, then we have $(\mathcal{C}\mathcal{E}f)(U) = \Lambda(\operatorname{epi} f \cap X \times (a,b))$ and $(\pi_p^2 \pi_*^2 \mathcal{C}\mathcal{E}f)(U) = \Lambda(\operatorname{epi} f \cap X \times [-\infty, b))$. Now let i be the inclusion of $\operatorname{epi} f \cap X \times [r,b)$ into $\operatorname{epi} f \cap X \times (a,b)$ and let j be the inclusion of $\operatorname{epi} f \cap X \times (a,b)$ into $\operatorname{epi} f \cap X \times [-\infty, b)$. First we prove that $\Lambda(i)$ is a bijection. Let $(x,y) \in \operatorname{epi} f \cap X \times (a,r)$, then $c\colon [0,1] \to \operatorname{epi} f \cap X \times (a,b), t \mapsto (x, t(r-y)+y)$ defines a path from (x,y) to (x,r), hence $\Lambda(i)$ is surjective. Now let $R\colon \operatorname{epi} f \cap X \times (a,b) \to \operatorname{epi} f \cap X \times [r,b), (x,y) \mapsto (x, \max\{r,y\})$, then $R \circ i = \mathrm{id}$ and thus $\Lambda(R) \circ \Lambda(i) = \mathrm{id}$, hence $\Lambda(i)$ is injective. By a similar argument we obtain that $\Lambda(j \circ i) = \Lambda(j) \circ \Lambda(i)$ is bijective. Now that both $\Lambda(i)$ and $\Lambda(j) \circ \Lambda(i)$ are bijective, we also have that $\Lambda(j)$ is bijective. But $\Lambda(j)$ is just $(\eta^{\pi^2} \circ \mathcal{C} \circ \mathcal{E})_{f\ U}$ and this implies the claim.

Now let $g\colon Y \to \mathbb{R}$ be another continuous function.

(20) **Corollary.** The interleavings of $\mathcal{C}\mathcal{E}f$ and $\mathcal{C}\mathcal{E}g$ are in bijection to those of $\pi_*^2 \mathcal{C}\mathcal{E}f$ and $\pi_*^2 \mathcal{C}\mathcal{E}g$.

A second type of Interleavings

We reuse the notation and the definitions from the previous subsection. In the present subsection we define the structure of a $-D$-category on $\mathbf{C}$ with the smoothing functor named $\mathcal{S}'$. Now for a smoothing functor we need two things, first we need the endofunctors $\mathcal{S}'(\mathbf{a})$ associated to any $\mathbf{a} \in -D$ and second we need the natural transformations $\mathcal{S}'(\mathbf{a} \preceq \mathbf{b})$ associated to any $\mathbf{a}, \mathbf{b} \in D$ with $\mathbf{a} \preceq \mathbf{b}$. Now in order to get there, we will do something seemingly unnecessary. We will define the endofunctors $\mathcal{S}'(\mathbf{a})$ not just on $\mathbf{D}$ but on the whole category of set-valued precosheaves on $\overline{D}$. Then we will show, that $\mathbf{C}$ is invariant under these endofunctors. In this sense the endofunctors $\mathcal{S}'(\mathbf{a})$ restrict to endofunctors on $\mathbf{C}$. Then we define the natural transformations.

So let us start with the endofunctors setting $\mathcal{S}'((a,b)) := S_*^{(-b,-b)}$ for all $(a,b) \in -D$. Now in order to see that $\mathbf{C}$ is invariant under these endofunctors we will show that π_*^2, π_p^2, η^{π^2}, and $\overline{\mathcal{S}}$ are compatible with these endofunctors in the same way they would by compatible if π_*^2 and π_p^2 were 1-homomorphisms and η^{π^2} was a 2-homomorphism. To this end we convince ourselves that the diagram

$$\begin{array}{ccccc}
\mathcal{Q} & \xrightarrow{(\pi^2)^{+1}} & \mathcal{U} & \xrightarrow{(\pi^2)^{-1}} & \mathcal{Q} \\
\big(S^{(-b,-b)}\big)^{-1} \big\downarrow & & \big\downarrow (s^{-b})^{-1} & & \big\downarrow \big(S^{(-b,-b)}\big)^{-1} \\
\mathcal{Q} & \xrightarrow[(\pi^2)^{+1}]{} & \mathcal{U} & \xrightarrow[(\pi^2)^{-1}]{} & \mathcal{Q}
\end{array}$$

commutes for all $b < \infty$.

Lemma. The functor π^2_* commutes with $\mathcal{S}'(\mathbf{a})$ and $\overline{\mathcal{S}}(\mathbf{a})$ for all $\mathbf{a} \in -D$.

Proof. This follows from the commutativity of the right square in the above diagram.

Lemma. The functor π^2_p commutes with $\overline{\mathcal{S}}(\mathbf{a})$ and $\mathcal{S}'(\mathbf{a})$ for all $\mathbf{a} \in -D$.

Proof. This follows from the commutativity of the left square in the above diagram.

Lemma. The natural transformation η^{π^2} commutes with $\mathcal{S}'(\mathbf{a})$ for all $\mathbf{a} \in -D$.

Proof. This follows from the commutativity of the outer square in the above diagram.

The previous three lemmata have the following

Corollary. The subcategory $\mathbf{C}$ is invariant under $\mathcal{S}'(\mathbf{a})$ for all $\mathbf{a} \in -D$.

So far we defined the endofunctors for the smoothing functor $\mathcal{S}'$ on $\mathbf{C}$. The next step is to define the natural transformations $\mathcal{S}'(\mathbf{a} \preceq \mathbf{b})$ for any $\mathbf{a}, \mathbf{b} \in D$ with $\mathbf{a} \preceq \mathbf{b}$. Now we already convinced ourselves that π^2_*, π^2_p, and η^{π^2} are compatible with the endofunctors of $\mathcal{S}'$ and $\overline{\mathcal{S}}$. So that is already half the part of π^2_* and π^2_p being 1-homomorphisms and η^{π^2} being a 2-homomorphism for this hypothetical smoothing functor. So it would be nice if we could define these natural transformations in such a way that π^2_*, π^2_p, and η^{π^2} were actually 1- respectively 2-homomorphisms. Now suppose we were already there, then we would have the commutative diagram

$$\begin{array}{ccc}
\mathcal{S}'(\mathbf{a}) & \xRightarrow{\mathcal{S}'(\mathbf{a}\preceq\mathbf{b})} & \mathcal{S}'(\mathbf{b}) \\
\eta^{\pi^2}\circ\mathcal{S}(\mathbf{a}) \Big\Downarrow & & \Big\Downarrow \eta^{\pi^2}\circ\mathcal{S}(\mathbf{b}) \\
\pi^2_p \circ \overline{\mathcal{S}}(\mathbf{a}) \circ \pi^2_* & \xRightarrow[\pi^2_p\circ\overline{\mathcal{S}}(\mathbf{a}\preceq\mathbf{b})\circ\pi^2_*]{} & \pi^2_p \circ \overline{\mathcal{S}}(\mathbf{b}) \circ \pi^2_*
\end{array}$$

for all $\mathbf{a}, \mathbf{b} \in D$ with $\mathbf{a} \preceq \mathbf{b}$. And this determines $\mathcal{S}'(\mathbf{a} \preceq \mathbf{b})$, since η^{π^2} is a natural isomorphism on $\mathbf{C}$. Also it does the job.

So now that π^2_* is a 1-homomorphism with respect to this second smoothing functor $\mathcal{S}'$ on $\mathbf{C}$ and $\overline{\mathcal{S}}$, the interleavings of $\mathcal{CE}f$ and $\mathcal{CE}g$ with respect to $\mathcal{S}'$ are in bijection with those of $\pi_*^2 \mathcal{C}\mathcal{E} f$ and $\pi_*^2 \mathcal{C}\mathcal{E} g$. Now these are all interleavings in $-D$-categories. But with $\overline{\mathcal{S}}$ we gave the category of set-valued precosheaves on $\overline{\mathbb{R}}_{-\infty}$ the structure of an $\overline{\mathbb{E}}$-category. So by lemma 18, from the section on complete persistence-enhancements, the interleavings of $\mathcal{C}\mathcal{E} f$ and $\mathcal{C}\mathcal{E} g$, with respect to the $-D$-category structure given by $\overline{\mathcal{S}}$, are in canonical bijection with those given by the $\overline{\mathcal{S}}$-induced structure of a D-category. So in conjunction with corollary 20 from the previous subsection we have the following

(21) **Propostion.** The interleavings of $\mathcal{C}\mathcal{E} f$ and $\mathcal{C}\mathcal{E} g$ with respect to $\mathcal{S}$ and with respect to $\mathcal{S}'$ are in bijection.

A negative Enhancement for Join Trees

To provide a negative persistence-enhancement for $\mathcal{R} \circ \mathcal{E}$ we first provide one for $\mathcal{E}$. To this end let $\mathbf{D}$ be the full subcategory of $\overline{\mathbb{R}}$-spaces of the form $r + \mathcal{E} f$ or $r + \mathcal{R}\mathcal{E} f$ for some bounded constructible $\mathbb{R}$-space f and some $r \in (-\infty, \infty]$. (We don't need constructibility for the results of this section, but we will need it later and we don't intend to introduce even more notation.) Now we define the smoothing functor $\mathcal{S}$ on $\mathbf{D}$. The easiest part of the definition are the endofunctors. Moreover these endofunctors exist on the whole category of $(-\infty, \infty]$-spaces and not just $\mathbf{D}$. Now let $f \colon X \to (-\infty, \infty]$ be a continuous function, then we set $\mathcal{S}((a, b))(f) := f - b$ for all $(a, b) \in -D$. And for any homomorphism φ in the category of $(-\infty, \infty]$-spaces we set $\mathcal{S}((a, b))(\varphi) := \varphi$. This defines the endofunctors $\{\mathcal{S}(\mathbf{a}) \mid \mathbf{a} \in -D\}$ on all $(-\infty, \infty]$-spaces.

Now let $f \colon X \to \mathbb{R}$ be an $\mathbb{R}$-space, let $r \in \mathbb{R}$, and let $(a, b), (c, d) \in -D$ with $(a, b) \preceq (c, d)$. Then we set

$$\mathcal{S}((a, b) \preceq (c, d))(r + \mathcal{E} f) \colon \operatorname{epi} f \to \operatorname{epi} f, (p, t) \mapsto (p, t - b + d).$$

Moreover we set

$$\mathcal{S}((a, b) \preceq (c, d))(r + \mathcal{R}\mathcal{E} f) := \mathcal{R}(\mathcal{S}((a, b) \preceq (c, d))(r + \mathcal{E} f)).$$

And this concludes our definition of $\mathcal{S}$.

Now for $(a, b) \in D^{\perp}$, the (a, b)-interleavings of two join trees are precisely the (a, b)-interleavings with respect to $\mathcal{S}$. In particular the interleaving distances coincide by corollary 12. Admittedly it is a bit of a cheat or kind of trivial, since we specifically defined this subcategory $\mathbf{D}$ so that this was going to work out.

Next we define the functor $\tilde{\mathcal{E}}$. To this end let $f \colon X \to \mathbb{R}$ and $g \colon Y \to \mathbb{R}$ be continuous functions, let $(a, b), (c, d) \in -D$, and let $\varphi \colon X \to Y$ be a homomorphism from $(f, (a, b))$ to $(g, (c, d))$ in $-\mathbf{F}$. Then we set $\tilde{\mathcal{E}}(f, (a, b)) := -b + \mathcal{E} f$, similarly for $(g, (c, d))$ of course, and $\tilde{\mathcal{E}}(\varphi) \colon \operatorname{epi} f \to \operatorname{epi} g, (p, t) \mapsto (\varphi(p), t - b + d)$. This defines our negative persistence-enhancement of $\mathcal{E}$. (Again a bit of a cheat.)

We note that we have the following

(22) **Lemma.** The endofunctor $\mathcal{R}$ is a 1-homomorphism from $\mathbf{D}$ to $\mathbf{D}$ and $\pi \colon \mathrm{id} \Rightarrow \mathcal{R}$ is a 2-homomorphism.

In particular $\mathcal{S}$ and $\mathcal{R} \circ \tilde{\mathcal{E}}$ form a negative persistence-enhancement of join trees. (This does not yield any new results about join trees, we just meant to show, they fit into this framework.)

Equality of Interleaving Distances

Finally we get to proving theorem 9. We reuse the notation and the definitions from the previous two subsections. We aim to show that the Reeb precosheaf $\mathcal{C}$ defines a 1-homomorphism from **D** to **C**. Now in order for this statement even to make any sense, the image of **D** under $\mathcal{C}$ should lie in **C**.

(23) **Lemma.** The image of **D** under $\mathcal{C}$ is part of the subcategory **C**.

Now let $f\colon X \to \mathbb{R}$ be a bounded constructible $\mathbb{R}$-space. To proof the previous lemma it suffices to show that $\mathcal{C}(r + \mathcal{E} f)$ and $\mathcal{C}(r + \mathcal{R}\mathcal{E} f)$ are an object of **C** for all $r \in (-\infty, \infty]$.

We consider the case $r = 0$ first. By lemma 19 the precosheaf $\mathcal{C}\mathcal{E} f$ is an object of **C**. By lemma 30 from the last appendix the projection $\mathcal{E} f$ from $\operatorname{epi} f$ to $\overline{\mathbb{R}}$ is a constructible $\overline{\mathbb{R}}$-space. This in conjunction with lemma 6 yields the following

(24) **Lemma.** The homomorphism $(\mathcal{C} \circ \pi \circ \mathcal{E})_f$ from $\mathcal{C}\mathcal{E} f$ to $\mathcal{C}\mathcal{R}\mathcal{E} f$ is an isomorphism of precosheaves.

Now **C** is closed under isomorphisms and thus also $\mathcal{C}\mathcal{R}\mathcal{E} f$ lies in **C**.

We continue with the case of r not necessarily being 0. To this end let $g\colon Y \to (-\infty, \infty]$ be a continuous function, possibly an object of **D**. We note that for any $r \in (-\infty, \infty]$ we have $r + g = \mathcal{S}((\infty, -r))(g)$, by definition of $\mathcal{S}$. We now recall that we also defined the endofunctors associated to the smoothing functor $\mathcal{S}'$ for **D** on the whole category of set-valued precosheaves on $\overline{D}$ and not just **D**. With **D** being invariant under these endofunctors, lemma 23 follows from the following

(25) **Lemma.** For all $\mathbf{a} \in -D$ we have $(\mathcal{C} \circ \mathcal{S}(\mathbf{a}))_g = (\mathcal{S}'(\mathbf{a}) \circ \mathcal{C})_g$.

Proof. For $(a, b) \in -D$ we have $\Delta \circ \mathcal{S}((a, b))(g) = S^{(-b,-b)} \circ \Delta \circ g$ and this implies the claim.

Next we show that $\mathcal{C}$ is compatible with the natural transformations of the smoothing functors. To this end let $\mathbf{a}, \mathbf{b} \in -D$ with $\mathbf{a} \preceq \mathbf{b}$.

(26) **Lemma.** We have $(\overline{\mathcal{S}}(\mathbf{a} \preceq \mathbf{b}) \circ \pi_*^2 \circ \mathcal{C} \circ \mathcal{E})_f = (\pi_*^2 \circ \mathcal{C} \circ \mathcal{S}(\mathbf{a} \preceq \mathbf{b}) \circ \mathcal{E})_f$.

Proof. First we set $(a', a) := \mathbf{a}$ and $(b', b) := \mathbf{b}$. Now let $r \in \mathbb{R}$. If we unravel the definitions we obtain

$$(\overline{\mathcal{S}}(\mathbf{a})\pi_*^2 \mathcal{C}\mathcal{E} f)([-\infty, r)) = \Lambda(\operatorname{epi} f \cap X \times [-\infty, r + a))$$

and
$$(\overline{\mathcal{S}}(\mathbf{b})\pi_*^2\mathcal{C}\mathcal{E}f)([-\infty, r)) = \Lambda(\operatorname{epi} f \cap X \times [-\infty, r+b)).$$
Let i be the inclusion of $\operatorname{epi} f \cap X \times [-\infty, r+a)$ into $\operatorname{epi} f \cap X \times [-\infty, r+b)$ and let

$$\tau\colon \operatorname{epi} f \cap X \times [-\infty, r+a) \to \operatorname{epi} f \cap X \times [-\infty, r+b), (p,t) \mapsto (p, t-a+b),$$

then
$$(\overline{\mathcal{S}}(\mathbf{a} \preceq \mathbf{b}) \circ \pi_*^2 \circ \mathcal{C} \circ \mathcal{E})_{f\ [-\infty,r)} = \Lambda(i)$$
and
$$(\pi_*^2 \circ \mathcal{C} \circ \mathcal{S}(\mathbf{a} \preceq \mathbf{b}) \circ \mathcal{E})_{f\ [-\infty,r)} = \Lambda(\tau).$$
Now let $(p,t) \in \operatorname{epi} f \cap X \times [-\infty, r+a)$, then $\{p\} \times [t, t-a+b]$ is contained in $\operatorname{epi} f \cap X \times [-\infty, r+b)$. Moreover we have $(p,t), (p, t-a+b) \in \{p\} \times [t, t-a+b]$, hence $\Lambda(i) = \Lambda(\tau)$.

Corollary. We have $(\mathcal{S}'(\mathbf{a} \preceq \mathbf{b}) \circ \mathcal{C} \circ \mathcal{E})_f = (\mathcal{C} \circ \mathcal{S}(\mathbf{a} \preceq \mathbf{b}) \circ \mathcal{E})_f$.

Proof. This follows from π_*^2 being a 1-homomorphism and being full and faithful on $\mathbf{C}$.

Corollary. We have $(\mathcal{S}'(\mathbf{a} \preceq \mathbf{b}) \circ \mathcal{C} \circ \mathcal{R} \circ \mathcal{E})_f = (\mathcal{C} \circ \mathcal{S}(\mathbf{a} \preceq \mathbf{b}) \circ \mathcal{R} \circ \mathcal{E})_f$.

Proof. By the lemmata 22 and 25 and the previous corollary we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{S}'(\mathbf{a})\mathcal{C}\mathcal{E}f & \xrightarrow{(\mathcal{S}'(\mathbf{a}\preceq\mathbf{b})\circ\mathcal{C}\circ\mathcal{E})_f} & \mathcal{S}'(\mathbf{b})\mathcal{C}\mathcal{E}f \\
\downarrow{\scriptstyle (\mathcal{S}'(\mathbf{a})\circ\mathcal{C}\circ\pi\circ\mathcal{E})_f} & & \downarrow{\scriptstyle (\mathcal{S}'(\mathbf{b})\circ\mathcal{C}\circ\pi\circ\mathcal{E})_f} \\
\mathcal{S}'(\mathbf{a})\mathcal{C}\mathcal{R}\mathcal{E}f & \xrightarrow[(\mathcal{C}\circ\mathcal{S}(\mathbf{a}\preceq\mathbf{b})\circ\mathcal{R}\circ\mathcal{E})_f]{} & \mathcal{S}'(\mathbf{b})\mathcal{C}\mathcal{R}\mathcal{E}f.
\end{array}$$

Now by lemma 24 the vertical arrows are isomorphisms and thus the naturality of $\mathcal{S}'(\mathbf{a} \preceq \mathbf{b})$ implies that $(\mathcal{C} \circ \mathcal{S}(\mathbf{a} \preceq \mathbf{b}) \circ \mathcal{R} \circ \mathcal{E})_f = (\mathcal{S}'(\mathbf{a} \preceq \mathbf{b}) \circ \mathcal{C} \circ \mathcal{R} \circ \mathcal{E})_f$.

Eventually the previous two corollaries and lemma 25 imply the

Proposition. The functor $\mathcal{C}$ is a 1-homomorphism from $\mathbf{D}$ to $\mathbf{C}$.

Now let $g\colon Y \to \mathbb{R}$ be another bounded constructible $\mathbb{R}$-space. Then we have the following

Corollary. The interleavings of $\mathcal{R}\mathcal{E}f$ and $\mathcal{R}\mathcal{E}g$ with respect to $\mathcal{S}$ are in bijection with the interleavings of $\mathcal{C}\mathcal{E}f$ and $\mathcal{C}\mathcal{E}g$ with respect to $\mathcal{S}'$.

Proof. We observe that all join trees of bounded constructible $\mathbb{R}$-spaces lie in a $-D$-subcategory of $\mathbf{D}$. By the previous proposition $\mathcal{C}$ yields a 1-homomorphism from this $-D$-category to $\mathbf{C}$. Moreover corollary 8 implies that $\mathcal{C}$ is full and faithful on this category and thus the interleavings of $\mathcal{RE}f$ and $\mathcal{RE}g$ are in bijection with those of $\mathcal{CRE}f$ and $\mathcal{CRE}g$. Now by lemma 24 these are isomorphic to $\mathcal{CE}f$ and $\mathcal{CE}g$ and this implies the claim.

Now we can finally proof theorem 9. To this end let $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ be continuous maps with X and Y smooth and compact manifolds.

Proof (Theorem 9). Let $\delta > 0$, by Bröcker and Jänich (1973 Satz 14.8) there are smooth functions $f'\colon X \to \mathbb{R}$ and $g'\colon Y \to \mathbb{R}$ with $\|f - f'\|_\infty \leq \frac{\delta}{8} \geq \|g - g'\|_\infty$. Further there are Morse functions $f''\colon X \to \mathbb{R}$ and $g''\colon Y \to \mathbb{R}$ with $\|f' - f''\|_\infty \leq \frac{\delta}{8} \geq \|g' - g''\|_\infty$ by Milnor (1963 corollary 6.8). These two results together yield $\|f - f''\|_\infty \leq \frac{\delta}{4} \geq \|g - g''\|_\infty$. By corollary 14, corollary 4, proposition 16, proposition 21, and the previous corollary, we have

$$\begin{aligned}
\mu(\mathcal{CE}f, \mathcal{CE}g) &\leq \mu(\mathcal{CE}f, \mathcal{CE}f'') + \mu(\mathcal{CE}f'', \mathcal{CE}g'') + \mu(\mathcal{CE}g'', \mathcal{CE}g) \\
&\leq \|f - f''\|_\infty + \mu_J(\mathcal{RE}f'', \mathcal{RE}g'') + \|g'' - g\|_\infty \\
&\leq \mu_J(\mathcal{RE}f'', \mathcal{RE}f) + \mu_J(\mathcal{RE}f, \mathcal{RE}g) + \mu_J(\mathcal{RE}g, \mathcal{RE}g'') + \frac{\delta}{2} \\
&\leq \|f'' - f\|_\infty + \mu_J(\mathcal{RE}f, \mathcal{RE}g) + \|g - g''\|_\infty + \frac{\delta}{2} \\
&\leq \mu_J(\mathcal{RE}f, \mathcal{RE}g) + \delta.
\end{aligned}$$

Similarly we have $\mu_J(\mathcal{RE}f, \mathcal{RE}g) \leq \mu(\mathcal{CE}f, \mathcal{CE}g) + \delta$. Since $\delta > 0$ was arbitrary, we have $\mu_J(\mathcal{RE}f, \mathcal{RE}g) = \mu(\mathcal{CE}f, \mathcal{CE}g)$.

The proof for M and M_J is completely analogous.

Relation to the Join Precosheaf

In the section Equivalence to descending Precosheaf we discussed the $\overline{\mathbb{E}}$-category of precosheaves on $\overline{\mathbb{R}}_{-\infty}$ with smoothing functor $\overline{\mathcal{S}}$ and the 1-homomorphism π^2_* from the D-category of precosheaves on $\overline{D}$. Together with $\mathcal{C}$ and $\tilde{\mathcal{C}}$ we obtain the functor $\pi^2_* \circ \mathcal{C}$ with the positive persistence-enhancement $\pi^2_* \circ \tilde{\mathcal{C}}$. And this is pretty much the precosheaf-theoretical version of the join tree introduced by Bubenik, de Silva, and Scott (2014 example 1.2.3). We name $\pi^2_* \circ \mathcal{C}$ the *join precosheaf*. With π^2_* being a 1-homomorphism we conclude from corollary 15 that the interleaving distances of join precosheaves provide lower bounds to the corresponding distances of Reeb precosheaves. Now we will see that this join precosheaf is indeed closely related to the other versions of join trees we have seen. We start with a self-contained description of a complete persistence-enhancement for $\pi^2_* \circ \mathcal{C}$ extending $\pi^2_* \circ \tilde{\mathcal{C}}$. To this end let $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ be continuous functions and let $(a, b), (c, d) \in \overline{\mathbb{E}}$. We set $(\widetilde{\pi^2_* \circ \mathcal{C}})((f, (a, b))) := (s^{-b} \circ \pi^2 \circ \Delta \circ f)_* \Lambda$. Now let $\varphi\colon (f, (a, b)) \to (g, (c, d))$ be a homomorphism in $\pm\mathbf{F}$ and let $U \subseteq \overline{\mathbb{R}}_{-\infty}$ be an open subset, then we have $\varphi((s^{-b} \circ \pi^2 \circ \Delta \circ f)^{-1}(U)) = \varphi((s^{-b} \circ f)^{-1}(U)) \subseteq \varphi((s^{-d} \circ f)^{-1}(U))$ and thus the restriction $\varphi|_{(s^{-b}\circ f)^{-1}(U)} \colon (s^{-b} \circ f)^{-1}(U) \to (s^{-d} \circ f)^{-1}(U)$. We set $\widetilde{(\pi_*^2 \circ \mathcal{C})}(\varphi)_U :=$ $\Lambda\big(\varphi|_{(s^{-b}\circ f)^{-1}(U)}\big)$.

Lemma. We have $\widetilde{(\pi_*^2 \circ \mathcal{C})}|_{\mathbf{F}} = \pi_*^2 \circ \tilde{C}$.

So the positive persistence-enhancement we get from $\widetilde{(\pi_*^2 \circ \mathcal{C})}$ coincides with $\pi_*^2 \circ \tilde{C}$. Now we examine the negative persistence-enhancement $\widetilde{(\pi_*^2 \circ \mathcal{C})}|_{(-\mathbf{F})}$ and how it relates to $\pi_*^2 \circ \mathcal{C} \circ \tilde{\mathcal{E}}$.

Lemma. The homomorphism $(\pi_*^2 \circ \mathcal{C} \circ \kappa)_f$ is an isomorphism from $(\pi_*^2 \circ \mathcal{C})(f)$ to $(\pi_*^2 \circ \mathcal{C} \circ \mathcal{E})(f)$.

The proof of this lemma is similar to the proof of lemma 19.

Corollary. The interleavings of $(\pi_*^2 \circ \mathcal{C})(f)$ and $(\pi_*^2 \circ \mathcal{C})(g)$ are in bijection to those of $(\pi_*^2 \circ \mathcal{C} \circ \mathcal{E})(f)$ and $(\pi_*^2 \circ \mathcal{C} \circ \mathcal{E})(g)$.

This corollary already captures what we care about most of the time, but we might also like to know whether the interleavings that we get from interleavings in $-\mathbf{F}$ are preserved by this bijection. An affirmative answer is provided by remark 17 and the following

Lemma. The functors $\overline{\mathcal{S}}$, $\widetilde{(\pi_*^2 \circ \mathcal{C})}|_{(-\mathbf{F})}$, and $\pi_*^2 \circ \mathcal{C} \circ \tilde{\mathcal{E}}$ combine to a negative persistence-enhancement for $\pi_*^2 \circ \mathcal{C} \circ \kappa$.

The proof of this lemma is similar to the proof of lemma 26.

Appendix

Constructible Spaces over the Reals

Definition. Let $S = \{a_1 < a_2 < \cdots < a_n\} \subset \mathbb{R}$ for some non-negative integer n and let $a_0 = -\infty$ and $a_{n+1} = \infty$.
Then an *S-skeleton X for an $\overline{\mathbb{R}}$-space* is given by the following data:

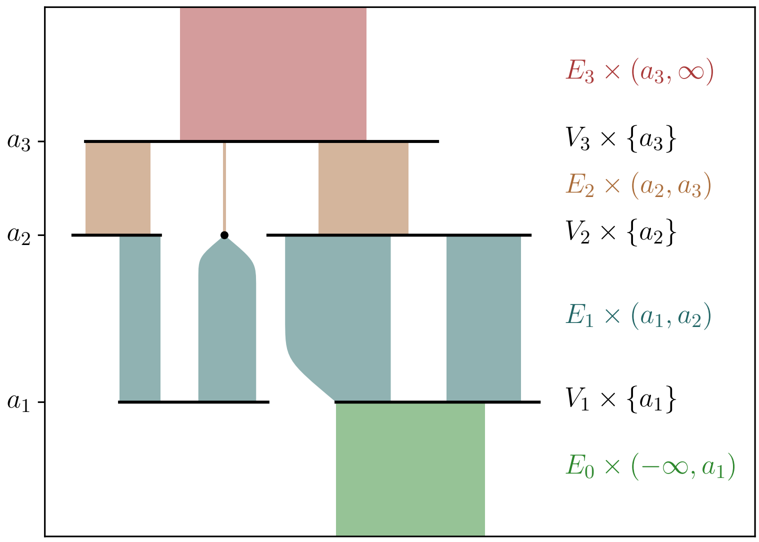
- For $i = 1, \ldots, n$ a locally path-connected compact topological space V_i.
- For $i = 0, \ldots, n$ a locally path-connected compact topological space E_i.
- For $i = 1, \ldots, n$ two continuous maps $l_i \colon E_i \to V_i$ and $r_{i-1} \colon E_{i-1} \to V_i$.

For an S-skeleton X as above we define the *geometric realization* $|X|$ to be

$$\left(\coprod_{i=1}^{n} V_i \times \{a_i\}\right) \coprod \coprod_{i=0}^{n} E_i \times [a_i, a_{i+1}] / \sim,$$

where $\sim$ is the equivalence relation generated by $(l_i(x), a_i) \sim (x, a_i)$ for $i = 1, \ldots, n$ and $(r_i(x), a_{i+1}) \sim (x, a_{i+1})$ for $i = 0, \ldots, n-1$. Moreover we define $f_X : |X| \to \overline{\mathbb{R}}$ to be the map induced by the projection onto the second factor.
An $\overline{\mathbb{R}}$-space given by a continuous map $g \colon Y \to \overline{\mathbb{R}}$ is *constructible* if there is a finite subset $S \subset \mathbb{R}$ and an S-skeleton X, such that $f_X \cong g$ as $\overline{\mathbb{R}}$-spaces.
The S-skeleton X is an *S-skeleton for a bounded $\mathbb{R}$-space* if $E_0 = \emptyset = E_n$.
A bounded $\mathbb{R}$-space $g \colon Y \to \mathbb{R}$ is *constructible* if there is a finite subset $S \subset \mathbb{R}$ and an S-skeleton X with $E_0 = \emptyset = E_n$, such that $f_X \cong g$ as $\overline{\mathbb{R}}$-spaces.

(27) *Example.* The following image depicts the geometric realization and the associated height function of an $\{a_1, a_2, a_3\}$-skeleton for an $\overline{\mathbb{R}}$-space.



Now let $S = \{a_1 < a_2 < \cdots < a_n\} \subset \mathbb{R}$ for some non-negative integer n and let $a_0 = -\infty$ and $a_{n+1} = \infty$.

(28) **Definition.** Let X and X' be two S-skeletons for an $\overline{\mathbb{R}}$-space and suppose we have the following data:

For $i = 1, \ldots, n$ a continuous map $\varphi_i^v \colon V_i \to V_i'$.
For $i = 0, \ldots, n$ a continuous map $\varphi_i^e \colon E_i \to E_i'$.

This data describes *a homomorphism of S-skeletons for an $\overline{\mathbb{R}}$-space* $\varphi \colon X \to X'$, if the two diagrams

$$\begin{array}{ccc} V_i & \xleftarrow{l_i} & E_i \\ \downarrow{\scriptstyle \varphi_i^v} & & \downarrow{\scriptstyle \varphi_i^e} \\ V_i' & \xleftarrow[l_i']{} & E_i' \end{array}$$

and

$$\begin{array}{ccc} E_{i-1} & \xrightarrow{r_{i-1}} & V_i \\ {\scriptstyle \varphi^e_{i-1}}\downarrow & & \downarrow{\scriptstyle \varphi^v_i} \\ E'_{i-1} & \xrightarrow[r'_{i-1}]{} & V'_i \end{array}$$

commute for $i = 1, \ldots, n$.
The composition of two homomorphisms of S-skeletons is defined by composing the individual maps φ_i^v and φ_i^e.

In the first definition we described the geometric realization of an S-skeleton for an $\overline{\mathbb{R}}$-space. This picture is only complete, if we also describe a geometric pendant to any homomorphism between S-skeletons.

Definition. For a homomorphism $\varphi\colon X \to X'$ of S-skeletons for $\overline{\mathbb{R}}$-spaces, we define the *geometric realization* $|\varphi|$ *of* φ to be the map induced on the quotients defined by the maps:

- $V_i \times \{a_i\} \to V_i' \times \{a_i\}, (p, a_i) \mapsto (\varphi_i^v(p), a_i)$ for $i = 1, \ldots, n$.
- $E_i \times [a_i, a_{i+1}] \to E_i' \times [a_i, a_{i+1}], (p, t) \mapsto (\varphi_i^e(p), t)$ for $i = 0, \ldots, n$.

Altogether this defines a faithful functor $|\_|$ from the category of S-skeletons for $\overline{\mathbb{R}}$-spaces to the category of $\overline{\mathbb{R}}$-spaces.

Graphs over the Reals

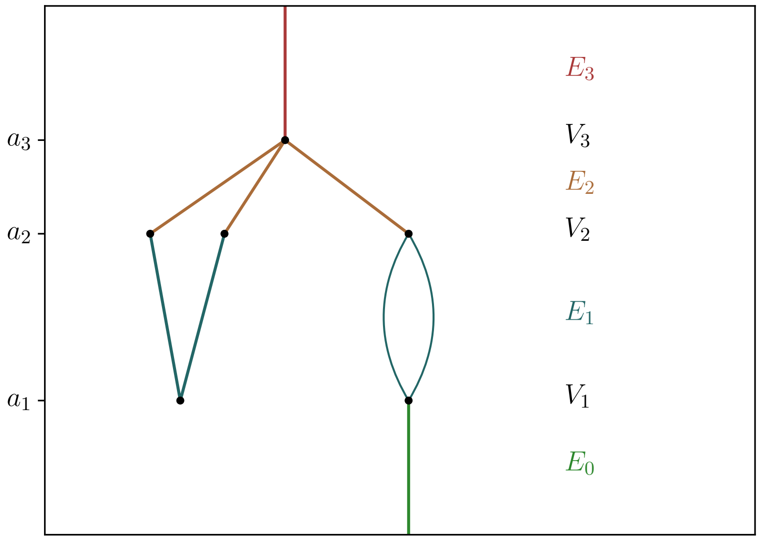
Definition. Let $S = \{a_1 < a_2 < \cdots < a_n\} \subset \mathbb{R}$ for some non-negative integer n. Then an S-graph G is given by the following data:

- For $i = 1, \ldots, n$ a finite set V_i.
- For $i = 0, \ldots, n$ a finite set E_i.
- For $i = 1, \ldots, n$ two maps $l_i\colon E_i \to V_i$ and $r_{i-1}\colon E_{i-1} \to V_i$.

Thinking of such sets V_i as vertices and the sets E_i as edges, G can (almost) be seen as a multigraph.

Definition. Let $S = \{a_1 < a_2 < \cdots < a_n\} \subset \mathbb{R}$ for some non-negative integer n and let X be an S-skeleton for an $\overline{\mathbb{R}}$-space, then we obtain an S-graph $\mathcal{C}X$ by applying the path-connected components functor π_0 to all spaces and maps defining X. Moreover we may apply π_0 to all the individual maps φ_i^v and φ_i^e describing a homomorphism φ of S-skeletons, to obtain a functor $\mathcal{C}$ from the category of S-skeletons to the category of S-graphs.

(29) *Example.* If X is the $\{a_1, a_2, a_3\}$-skeleton depicted in example 27, then the following image shows $\mathcal{C}X$.



Now let $S = \{a_1 < a_2 < \cdots < a_n\} \subset \mathbb{R}$ for some non-negative integer n. The definition of a homomorphism between S-graphs is completely analogous to definition 28, homomorphisms between S-skeletons for $\overline{\mathbb{R}}$-spaces. More precisely the full subcategory of the category of S-skeletons for $\overline{\mathbb{R}}$-spaces with all V_i and E_i discrete is isomorphic to the category of S-graphs. For convenience we spell out the details nevertheless.

Definition. Let G and G' be two S-graphs and suppose we have the following data:

- For $i = 1, \dots, n$ a map $\varphi_i^v \colon V_i \to V_i'$.
- For $i = 0, \dots, n$ a map $\varphi_i^e \colon E_i \to E_i'$.

This data describes *a homomorphism of S-graphs* $\varphi \colon G \to G'$, if the two diagrams

$$\begin{array}{ccc} V_i & \xleftarrow{l_i} & E_i \\ \downarrow{\scriptstyle \varphi_i^v} & & \downarrow{\scriptstyle \varphi_i^e} \\ V_i' & \xleftarrow[l_i']{} & E_i' \end{array}$$

and

$$\begin{array}{ccc} E_{i-1} & \xrightarrow{r_{i-1}} & V_i \\ \downarrow{\scriptstyle \varphi_{i-1}^e} & & \downarrow{\scriptstyle \varphi_i^v} \\ E_{i-1}' & \xrightarrow[r_{i-1}']{} & V_i' \end{array}$$

commute for $i = 1, \dots, n$.

The composition of two homomorphisms of S-graphs is defined by composing the individual maps φ_i^v and φ_i^e.

The functor $|\_|$ defined on the category of S-skeletons for $\overline{\mathbb{R}}$-spaces from the previous section restricts to the category of S-graphs, if we identify the category of S-graphs with the full subcategory of the category of S-skeletons for $\overline{\mathbb{R}}$-spaces with all V_i and E_i discrete.

Lemma. The induced functor $|\_|$ on S-graphs as described above is full and faithful.

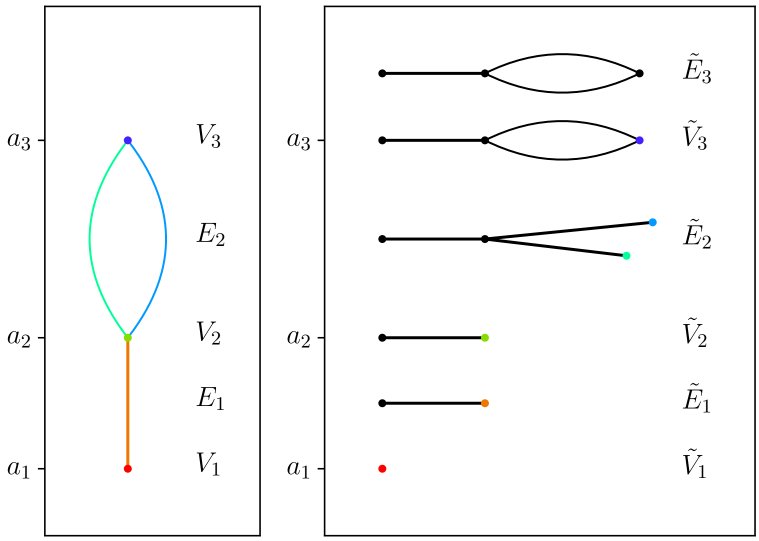
A Skeleton for the Epigraph

Let $S = \{a_1 < a_2 < \cdots < a_n\} \subset \mathbb{R}$ for some non-negative integer n and let X be an S-skeleton for a bounded $\mathbb{R}$-space.

Definition. The *epigraph* $\operatorname{epi} X$ *of* X is the S-skeleton defined by

- $\tilde{V}_i := f_X^{-1}((-\infty, a_i])$ for $i = 1, \ldots, n$,
- $\tilde{E}_0 = \emptyset$, $\tilde{E}_i := f_X^{-1}((-\infty, a_i]) \coprod (E_i \times [a_i, a_{i+1}])/(l_i(x), a_i) \sim (x, a_i)$ for $i = 1, \ldots, n-1$, and $\tilde{E}_n := |X|$, and
- $\tilde{l}_i := \left[\mathrm{id} \coprod l_i \circ \pi^1\right]$ and $\tilde{r}_{i-1}$ the canonical quotient map for $i = 1, \ldots, n$.

Example. The left graphic in the following picture shows an $\{a_1, a_2, a_3\}$-graph, which we view as an $\{a_1, a_2, a_3\}$-skeleton for a bounded $\mathbb{R}$-space, by augmenting the sets of vertices and edges with the discrete topology.



We ignore the colors for now. The right graphic shows the corresponding epigraph.

Moreover there is a natural homomorphism from X to $\operatorname{epi} X$.

Definition. We define the homomorphism $\kappa_X \colon X \to \operatorname{epi} X$ by the following data:

- $\kappa^v_{Xi} \colon V_i \to \tilde{V}_i, p \mapsto (p, a_i)$ for $i = 1, \dots, n$.
- $\kappa^e_{Xi} \colon E_i \to \tilde{E}_i, p \mapsto (p, a_{i+1})$ for $i = 1, \dots, n-1$.

Example. In the previous example each vertex and edge of the $\{a_1, a_2, a_3\}$-graph is depicted in a different color. Now κ_X maps each vertex or edge to the point of the same color in the epigraph.

Now we show that $f_{\operatorname{epi} X}$ and $\mathcal{E} f_X$ are naturally isomorphic.

Definition. We define a continuous map φ_X from $|\operatorname{epi} X|$ to the epigraph $\operatorname{epi} f_X$ of the continuous map f_X. In some sense most of $|\operatorname{epi} X|$ can already be seen as a subset of $\operatorname{epi} f_X$ and on this part we choose φ_X to be the inclusion. The part where this does not work is $(V_i \coprod (E_i \times [a_i, a_{i+1}])/(l_i(x), a_i) \sim (x, a_i)) \times [a_i, a_{i+1}]/\sim$ and on this part we define φ_X to be $(p, x, y) \mapsto \left(p, a_i + (x - a_i)\frac{y - a_i}{a_{i+1} - a_i}, y\right)$.

(30) **Lemma.** The map φ_X is a homeomorphism and we have $f_{|\operatorname{epi} X|} = \pi^2 \circ \varphi_X = \mathcal{E} f_X \circ \varphi_X$.

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